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GMM Gradient Tests for Spatial Dynamic Panel Data Models *

Süleyman Taşpınar† Osman Doğan‡ Anil K. Bera§

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Abstract

In this study, we formulate the adjusted gradient tests when the alternative model used to construct tests deviates from the true data generating process for a spatial dynamic panel data model (SDPD). Following Bera et al. (2010), we introduce these adjusted gradient tests along with the standard ones within a GMM framework. These tests can be used to detect the presence of (i) the contemporaneous spatial lag terms, (ii) the time lag term, and (iii) the spatial time lag terms in an higher order SDPD model. These adjusted tests have two advantages: (i) their null asymptotic distribution is a central chi-squared distribution irrespective of the mis-specified alternative model, and (ii) their test statistics are computationally simple and require only the ordinary least-squares (OLS) estimates from a non-spatial two-way panel data model. We investigate the finite sample size and power properties of these tests through Monte Carlo studies. Our results indicates that the adjusted gradient tests have good finite sample properties.

JEL-Classification: C13, C21, C31.
Keywords: Spatial Dynamic Panel Data Model, SDPD, GMM, Robust LM Tests, GMM Gradient Tests, Inference.

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1 Introduction

In this study, we consider a spatial dynamic panel data model (SDPD) that includes a time lag term, spatial time lag terms and contemporaneous spatial lag terms. The model is in the form of a high order spatial autoregressive model by including high orders of contemporaneous spatial lag term and spatial time lag term. We formulate the GMM gradient tests, the adjusted GMM gradient tests and the $C(\alpha)$ test to test hypothesis about the parameters of the time lag term, the spatial time lag terms and the contemporaneous spatial lag terms.

In the literature, the model specifications and estimation strategies, including the ML, GMM and Bayesian methods, receive considerably more attention than the specification testing and other forms of hypothesis tests for the SDPD models. For two recent surveys, see Anselin et al. (2008) and Lee and Yu (2010b). Lee and Yu (2010a, 2011, 2012a), Yu and Lee (2010), and Yu et al. (2008, 2012) consider the ML approach for dynamic spatial panel data models when both the number of individuals and the number of time periods are large under various scenarios. The MLE suggested in these studies has asymptotic bias and the limiting distributions of bias corrected versions are properly centered when the number of time periods grows faster than the number of individuals. Elhorst (2005), Lee and Yu (2015), and Su and Yang (2015) consider the ML approach for the dynamic panel data models that have spatial autoregressive processes in the disturbance terms. Parent and LeSage (2011) introduce the Bayesian MCMC method for a panel data model that accommodates dependence across space and time in the error components. Kapoor et al. (2007) extend the GMM approach of Kelejian and Prucha (2010) to a static spatial panel data model with error components. Lee and Yu (2014) consider the GMM approach for an SDPD model that has high orders of contemporaneous spatial lag term and spatial time lag term.

To date, the focus has been on the specification testing for the cross-sectional and the static spatial panel data models (Anselin et al. 1996; Baltagi and Yang 2013; Baltagi et al. 2003, 2007; Debarsy and Ertur 2010). In this study, we introduce GMM-based tests for an SDPD model that has high orders of contemporaneous spatial lag term and spatial time lag term. In particular, we first consider the GMM-gredient test (or the LM test) of Newey and West (1987), which can be used to test the non-linear restrictions on the parameter vector. We also consider the $C(\alpha)$ test within the GMM framework for the same model. While the computation of GMM-gradient test requires an estimate of the optimal restricted GMME, the computation of $C(\alpha)$ test statistic requires only a consistent estimate of the parameter vector. For both tests, we provide analytical justification for their asymptotic distributions within the context of our SDPD.

Within the ML framework, Davidson and MacKinnon (1987), Saikkonen (1989) and Bera and Yoon (1993) show that the usual LM tests are not robust to local mis-specifications in the alternative models. That is, the usual LM tests have non-central chi-squared distribution when the alternative model (locally) deviates from the true data generating process. Bera et al. (2010) extent this result to the GMM framework and show that the asymptotic distribution of the usual GMM-gradient test is a non-central chi-squared distribution when the alternative model deviates from the true data generating process. In such a context, the usual LM and GMM-gradient tests will over reject the true null hypothesis. Therefore, Bera and Yoon (1993) and Bera et al. (2010) suggest robust (or adjusted) versions that have, asymptotically, central chi-squared distributions irrespective of the local deviations of the alternative models from the true data generating process.

By following Bera et al. (2010), we construct various adjusted GMM-gradient tests for an SDPD model. These tests can be used to detect the presence of (i) the spatial lag terms, (ii) the time lag term, and (iii) the spatial time lag terms in an SDPD model. Besides being robust to local mis-specifications, these tests are computationally simple and require only estimates from a non-spatial two-way panel data model. Within the context of our SDPD, we analytically show the asymptotic distribution of robust tests under both the null and local alternative hypotheses. We investigate the size and power properties of our suggested robust tests through a Monte Carlo simulation. The simulation results are in line with our theoretical findings and indicate that the robust tests have good size and power properties.

The rest of this paper is organized in the following way. Section 2 presents the SDPD model under consideration and discusses its assumptions. Section 3 lays out the details of the GMM estimation approach for the model specification. Section 4 presents the GMM gradient tests, the adjusted GMM gradient tests and the $C(\alpha)$ test. Section 5 lays out the details of the Monte Carlo design and presents the results. Section 6 closes with concluding remarks. Some of the technical derivations are relegated to an appendix.

2 The Model Specification and Assumptions

Using the standard notation, an SDPD model with both individual and time fixed effects is stated as

$$Y_{nt} = \sum_{j=1}^{p} \lambda_{j0} W_{nj} Y_{nt} + \gamma_0 Y_{n,t-1} + \sum_{j=1}^{p} \rho_{j0} W_{nj} Y_{n,t-1} + X_{nt}\beta_0 + \mathbf{c}_{n0} + \alpha_{t0} l_n + V_{nt} \tag{2.1}$$

for $t = 1, 2, \ldots, T$, where $Y_{nt} = (y_{1t}, y_{2t}, \ldots, y_{nt})^{'}$ is the $n \times 1$ vector of a dependent variable, X_n is the $n \times k_x$ matrix of non-stochastic exogenous variables with a matching parameter vector β_0, and $V_{nt} = (v_{1t}, \ldots, v_{nt})^{'}$ is the $n \times 1$ vector of disturbances (or innovations). The spatial lags of the dependent variable at time t and $t-1$ are, respectively, denoted by $W_{nj}Y_{nt}$ and $W_{nj}Y_{n,t-1}$ for $j = 1, \ldots, p$. Here, W_{nj}s are the $n \times n$ spatial weight matrices of known constants with zero diagonal elements, $\lambda_0 = (\lambda_{10}, \ldots, \lambda_{p0})^{'}$ and $\rho_0 = (\rho_{10}, \ldots, \rho_{p0})^{'}$ are the spatial autoregressive parameters. The individual fixed effects are denoted by $\mathbf{c}_{n0} = (c_{1,0}, \ldots, c_{n,0})^{'}$ and the time fixed effect is denoted by $\alpha_{t0} l_n$, where l_n is the $n \times 1$ vectors of ones. For the identification of fixed effects, Lee and Yu (2014) impose the normalization $l_n^{'}\mathbf{c}_{n0} = 0$. For the estimation of the model, we assume that Y_{n0} is observable. Let Θ be the parameter space of the model. In order to distinguish the true parameter vector from other possible values in Θ, we state the model with the true parameter vector $\theta_0 = \left(\lambda_0^{'}, \delta_0^{'}\right)^{'}$, where $\delta_0 = \left(\gamma_0, \rho_0^{'}, \beta_0^{'}\right)^{'}$. Furthermore, for notational simplicity we let $S_n(\lambda) = \left(I_n - \sum_{j=1}^{p} \lambda_j W_{nj}\right)$, $S_n = S_n(\lambda_0)$, $A_n = S_n^{-1}\left(\gamma_0 I_n + \sum_{j=1}^{p} \rho_j W_{nj}\right)$, $G_{nj}(\lambda) = W_{nj} S_n^{-1}(\lambda)$, $G_{nj} = G_{nj}(\lambda_0)$ and $N = n(T-1)$.

To avoid the incidental parameter problem, the model is transformed to wipe out the fixed effects. The individual effects can be eliminated from the model by employing the orthonormal eigenvector matrix $\left[F_{T,T-1}, \frac{1}{\sqrt{T}} l_T\right]$ of $J_T = \left(I_T - \frac{1}{T} l_T l_T^{'}\right)$, where $F_{T,T-1}$ is the $T \times (T-1)$ eigenvectors matrix corresponding to the eigenvalue one and l_T is the $T \times 1$ vector of ones corresponding to the eigenvalue zero.[1] This orthonormal transformation can be applied by writing the model in an $n \times T$ system. Hence, the dependent variable is transformed as $\left[Y_{n1}, Y_{n2}, \ldots, Y_{nT}\right] \times F_{T,T-1} = \left[Y_{n1}^{*}, Y_{n2}^{*}, \ldots, Y_{n,T-1}^{*}\right]$, and also $\left[Y_{n0}, Y_{n1}, \ldots, Y_{n,T-1}\right] \times F_{T,T-1} = \left[Y_{n0}^{(*,-1)}, Y_{n1}^{(*,-1)}, \ldots, Y_{n,T-2}^{(*,-1)}\right]$. Similarly, $\left[X_{nj,1}, X_{nj,2}, \ldots, X_{nj,T}\right] \times F_{T,T-1} = \left[X_{nj,1}^{*}, X_{nj,2}^{*}, \ldots, X_{nj,T-1}^{*}\right]$ for $j = 1, \ldots, k_x$, $\left[V_{n1}, V_{n2}, \ldots, V_{nT}\right] \times F_{T,T-1} = \left[V_{n1}^{*}, V_{n2}^{*}, \ldots, V_{n,T-1}^{*}\right]$, and $\left[\alpha_{10}, \alpha_{20}, \ldots, \alpha_{T0}\right] \times F_{T,T-1} = \left[\alpha_{10}^{*}, \alpha_{20}^{*}, \ldots, \alpha_{T-1,0}^{*}\right]$. Since the column of $\left[F_{T,T-1}, \frac{1}{\sqrt{T}} l_T\right]$ are orthonormal, we have $[\mathbf{c}_{n0}, \mathbf{c}_{n0}, \ldots, \mathbf{c}_{n0}] \times F_{T,T-1} = 0_{n\times(T-1)}$. Thus, the transformed model does

[1] This orthonormal matrix has the following properties (i) $J_T F_{T,T-1} = F_{T,T-1}$ and $J_T l_T = 0_{T\times 1}$, (ii) $F_{T,T-1}^{'} F_{T,T-1} = I_{T-1}$ and $F_{T,T-1}^{'} l_T = 0_{(T-1)\times 1}$, (iii) $F_{T,T-1} F_{T,T-1}^{'} + \frac{1}{T} l_T l_T^{'} = I_T$ and (iv) $F_{T,T-1} F_{T,T-1}^{'} = J_T$.

not include the individual fixed effects and can be written as

$$Y_{nt}^* = \sum_{j=1}^{p} \lambda_{j0} W_{nj} Y_{nt}^* + \gamma_0 Y_{n,t-1}^{(*,-1)} + \sum_{j=1}^{p} \rho_{j0} W_{nj} Y_{n,t-1}^{(*,-1)} + X_{nt}^* \beta_0 + \alpha_{t0}^* l_n + V_{nt}^* \tag{2.2}$$

for $t = 1, \ldots, T-1$. We consider the forward orthogonal difference (FOD) transformation for the orthonormal transformation. Hence, the terms in (2.2) can be explicitly stated as $V_{nt}^* = \left(\frac{T-t}{T-t+1}\right)^{1/2} \left[V_{nt} - \frac{1}{T-t} \sum_{h=t+1}^{T} V_{nh}\right]$, $Y_{n,t-1}^{(*,-1)} = \left(\frac{T-t}{T-t+1}\right)^{1/2} \left[Y_{n,t-1} - \frac{1}{T-t} \sum_{h=t}^{T-1} Y_{nh}\right]$, and the others terms are defined similarly. Let $\mathbf{V}_{n,T-1}^* = \left(V_{n1}^{*'}, \ldots, V_{n,T-1}^{*'}\right)^{'}$. Then, $\mathrm{Var}\left(\mathbf{V}_{n,T-1}^*\right) = \left(F_{T,T-1}^{'} \otimes I_n\right) \mathrm{E}\left(\mathbf{V}_{nT} \mathbf{V}_{nT}^{'}\right) \left(F_{T,T-1} \otimes I_n\right) = \sigma_0^2 I_N$ by Assumption 1. The transformed model in (2.2) still includes the time fixed effect $\alpha_{t0}^* l_n$, which can be eliminated by pre-multiplying the model with $J_n = I_n - \frac{1}{n} l_n l_n^{'}$. The resulting model is free of the fixed effects, for $t = 1, \ldots, T-1$,

$$J_n Y_{nt}^* = \sum_{j=1}^{p} \lambda_{j0} J_n W_{nj} Y_{nt}^* + \gamma_0 J_n Y_{n,t-1}^{(*,-1)} + \sum_{j=1}^{p} \rho_{j0} J_n W_{nj} Y_{n,t-1}^{(*,-1)} + J_n X_{nt}^* \beta_0 + J_n V_{nt}^*. \tag{2.3}$$

The consistency and asymptotic normality of the GMME of θ_0 are established under Assumptions 1 through 5.[2]

Assumption 1. — The innovations v_{it}s are independently and identically distributed across i and t, and satisfy $\mathrm{E}\left(v_{it}\right) = 0$, $\mathrm{E}\left(v_{it}^2\right) = \sigma_0^2$, and $\mathrm{E}\left|v_{it}\right|^{4+\eta} < \infty$ for some $\eta > 0$ for all i and t.

Assumption 2. — The spatial weight matrix W_{nj}s is uniformly bounded in row and column sums in absolute value for $j = 1, \ldots, p$, and $\| \sum_{j=1}^{p} \lambda_{j0} W_{nj} \|_\infty < 1$. Moreover, $S_n^{-1}(\lambda)$ exists and is uniformly bounded in row and column sums in absolute value for all values of λ in a compact parameter space.

Assumption 3. — Let $\eta > 0$ be a real number. Assume that X_{nt}, $\mathbf{c_{n0}}$, and α_{t0} are non-stochastic terms satisfying (i) $\sup_{n,T} \frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \left|x_{it,l}\right|^{2+\eta} < \infty$ for $l = 1, \ldots, k_x$, where $x_{it,l}$ is the (i,t)th element of the lth column, (ii) $\lim_{n\to\infty} \frac{1}{n(T-1)} \sum_{t=1}^{T-1} X_{nt}^* J_n X_{nt}^*$ exists and is non-singular, and (iii) $\sup_T \frac{1}{T} \sum_{t=1}^{T} \left|\alpha_{t0}\right|^{2+\eta} < \infty$ and $\sup_n \frac{1}{n} \sum_{i=1}^{n} \left|c_{i0}\right|^{2+\eta} < \infty$.

Assumption 4. — The DGP for the initial observations is $Y_{n0} = \sum_{h=0}^{h^*} A_n^h S_n^{-1} \left(\mathbf{c}_{n0} + X_{n,-h} \beta_0 + \alpha_{-h,0} l_n + V_{n,-h}\right)$, where h^* could be finite or infinite.

Assumption 5. — The elements of $\sum_{h=0}^{\infty} \mathrm{abs}\left(A_n^h\right)$ are uniformly bounded in row and column sums in absolute value, where $\left[\mathrm{abs}\left(A_n\right)\right]_{ij} = \left|A_{n,ij}\right|$

3 The GMM Estimation Approach

In this section, we summarize the GMM estimation approach for (2.3) under both large T and finite T scenarios. The model in (2.3) indicates that IVs are needed for $W_{nj} Y_{nt}^*$, $Y_{n,t-1}^{(*,-1)}$, and $W_{nj} Y_{n,t-1}^{(*,-1)}$ for each t. Before, we introduce the set of moment functions, it will be convenient to introduce some further notations. Let $Z_{nt}^* = \left[Y_{n,t-1}^{(*,-1)}, W_{n1} Y_{n,t-1}^{(*,-1)}, \ldots, W_{np} Y_{n,t-1}^{(*,-1)}, X_{nt}^*\right]$, $\mathbf{J}_{n,T-1} = I_{T-1} \otimes J_n$, and $\mathbf{V}_{n,T-1}^*(\theta) = \left(V_{n1}^{*'}(\theta), \ldots, V_{n,T-1}^{*'}(\theta)\right)^{'}$ where $V_{nt}^*(\theta) = S_{nt}(\lambda) Y_{nt}^* - Z_{nt}^* \delta - \alpha_t^* l_n$. We consider the

[2]For interpretations and implications of these assumptions, see Lee and Yu (2014) and Kelejian and Prucha (2010).

following $(m+q)\times 1$ vector of moment functions

$$g_{nT}(\theta)=\begin{pmatrix}\mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}_{n1,T-1}\mathbf{J}_{n,T-1}\mathbf{V}^{*}_{n,T-1}(\theta)\\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}_{n2,T-1}\mathbf{J}_{n,T-1}\mathbf{V}^{*}_{n,T-1}(\theta)\\ \vdots\\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}_{nm,T-1}\mathbf{J}_{n,T-1}\mathbf{V}^{*}_{n,T-1}(\theta)\\ \mathbf{Q}^{'}_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{V}^{*}_{n,T-1}(\theta)\end{pmatrix}. \tag{3.1}$$

In (3.1), $\mathbf{P}_{nj,T-1}=I_{T-1}\otimes P_{nj}$, where P_{nj} is the $n\times n$ quadratic moment matrix satisfying $\operatorname{tr}(P_{nj}J_n)=0$ for $j=1,\ldots,m$, and $\mathbf{Q}_{n,T-1}=\left(Q^{'}_{n1},\ldots,Q^{'}_{n,T-1}\right)^{'}$ is the $N\times q$ liner IV matrix such that $q\geq k_x+2p+1$. Under Assumptions 1-4, it can be shown that $\frac{1}{N}\frac{\partial g_{nT}(\theta_0)}{\partial\theta'}=D_{nT}+R_{nT}+O\left(\frac{1}{\sqrt{nT}}\right)$, where D_{nT} is $O(1)$ and R_{nT} is $O\left(\frac{1}{T}\right)$.[3]

Let $\operatorname{vec}_D(\cdot)$ be the operator that creates a column vector from the diagonal elements of an input square matrix. For the optimal GMM estimation, we need to calculate the covariance matrix of moment functions $\mathrm{E}\left(g^{'}_{nT}(\theta_0)\,g_{nT}(\theta_0)\right)$, which can be approximated by

$$\begin{aligned}\Sigma_{nT}=\sigma_0^4&\begin{pmatrix}\frac{1}{N}\Delta_{nm,T} & 0_{m\times q}\\ 0_{q\times m} & \frac{1}{\sigma_0^2}\frac{1}{N}\mathbf{Q}^{'}_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{Q}_{n,T-1}\end{pmatrix}\\ &+\frac{1}{N}\begin{pmatrix}\left(\mu_4-3\sigma_0^4\right)\omega^{'}_{nm,T}\omega_{nm,T} & 0_{m\times q}\\ 0_{q\times m} & 0_{q\times m}\end{pmatrix},\end{aligned} \tag{3.2}$$

where $\omega_{nm,T}=\left[\operatorname{vec}_D(\mathbf{J}_{n,T-1}\mathbf{P}_{n1,T-1}\mathbf{J}_{n,T-1}),\ldots,\operatorname{vec}_D(\mathbf{J}_{n,T-1}\mathbf{P}_{nm,T-1}\mathbf{J}_{n,T-1})\right]$, $\Delta_{nm,T}=\left[\operatorname{vec}(\mathbf{J}_{n,T-1}\mathbf{P}^{'}_{n1,T-1}\mathbf{J}_{n,T-1}),\ldots,\operatorname{vec}(\mathbf{J}_{n,T-1}\mathbf{P}^{'}_{nm,T-1}\mathbf{J}_{n,T-1})\right]^{'}\times\left[\operatorname{vec}(\mathbf{J}_{n,T-1}\mathbf{P}^{s}_{n1,T-1}\mathbf{J}_{n,T-1}),\ldots,\operatorname{vec}(\mathbf{J}_{n,T-1}\mathbf{P}^{s}_{nm,T-1}\mathbf{J}_{n,T-1})\right]$, where $A^s_n=A_n+A^{'}_n$ for any square matrix A_n.

Let $\widehat{\Sigma}_{nT}$ be a consistent estimate of Σ_{nT}. Then, the optimal GMME is defined by

$$\widehat{\theta}_{nT}=\operatorname{argmin}_{\theta\in\Theta}\,g^{'}_{nT}(\theta)\,\widehat{\Sigma}^{-1}_{nT}g_{nT}(\theta) \tag{3.3}$$

Under Assumptions 1 - 5, Lee and Yu (2014) show that when both T and n tend to infinity[4]:

$$\sqrt{N}\left(\widehat{\theta}_{nT}-\theta_0\right)\xrightarrow{d}N\left(0,\,\left[\operatorname{plim}_{n,T\to\infty}D^{'}_{nT}\Sigma^{-1}_{nT}D_{nT}\right]^{-1}\right). \tag{3.4}$$

When T is finite, the GMME in (3.4) is still consistent and unbiased but its limiting covariance matrix is different, since the additional term $R_{nT}=O\left(\frac{1}{T}\right)$ does not vanish. Hence, when T is finite, the asymptotic covariance matrix of $\sqrt{N}\left(\widehat{\theta}_{nT}-\theta_0\right)$ is given by $\left[\operatorname{plim}_{n\to\infty}\left(D_{nT}+R_{nT}\right)^{'}\Sigma^{-1}_{nT}\left(D_{nT}+R_{nT}\right)\right]^{-1}$.

[3] The explicit forms for D_{nT} and R_{nT} are not required for our testing results, hence they are not given here. For these terms, see Lee and Yu (2014).

[4] Lee and Yu (2014) state the identification conditions. Here, we simply assume that the parameter vector is identified.

4 The GMM Gradient Tests

In this section, we consider various version of the gradient test (LM test). Let $r : \mathbb{R}^{2p+k_x+1} \to \mathbb{R}^{k_r}$ be a twice continuously differentiable function, and assume that $R(\theta) = \frac{\partial r(\theta)}{\partial \theta'}$ has rank k_r. Consider the implicit restrictions denoted by the null hypothesis $H_0 : r(\theta_0) = 0$. Define $\widehat{\theta}_{nT,r} = \text{argmax}_{\{\theta: r(\theta)=0\}} \mathcal{Q}_n$, where $\mathcal{Q}_n = g'_{nT}(\theta)\,\widehat{\Sigma}^{-1}_{nT} g_{nT}(\theta)$, as a restricted (or constrained) optimal GMME.

In order to give a general argument, consider the following partition of $\theta = \left(\beta', \psi', \phi'\right)'$, where ψ and ϕ are, respectively, $k_\psi \times 1$ and $k_\phi \times 1$ vectors such that $k_\psi + k_\phi = 2p + 1$. In the context of our model, ψ and ϕ can be any combinations of the remaining parameters, namely, $\left(\lambda', \gamma, \rho'\right)'$. Let $G_a = \frac{1}{N}\frac{\partial g_{nT}(\theta)}{\partial a'}$, $C_a = G'_a(\theta)\,\widehat{\Sigma}^{-1}_{nT}\bar{g}_{nT}(\theta)$, where $a \in \{\beta, \psi, \phi\}$ and $\bar{g}_{nT} = \frac{1}{N} g_{nT}$. Define $G(\theta) = \left(G_\beta(\theta), G_\psi(\theta), G_\phi(\theta)\right)$, and $C(\theta) = \left(C'_\beta(\theta), C'_\psi(\theta), C'_\phi(\theta)\right)'$, and $B(\theta) = G'(\theta)\,\widehat{\Sigma}^{-1}_{nT} G(\theta)$. Finally, let $\mathcal{G}_a = \text{plim}_{n,T\to\infty} \frac{1}{N}\frac{\partial g_{nT}(\theta_0)}{\partial a'}$ for $a \in \{\beta, \psi, \phi\}$. Define $\mathcal{G} = \left(\mathcal{G}_\beta, \mathcal{G}_\psi, \mathcal{G}_\phi\right)$ and $\mathcal{H} = \text{plim}_{n,T\to\infty} \left(D_{nT} + R_{nt}\right)' \widehat{\Sigma}^{-1}_{nT}\left(D_{nT} + R_{nt}\right)$. We consider the following partition of $B(\theta)$ and $\mathcal{H}$:

$$B(\theta) = \begin{pmatrix} B_\beta(\theta) & B_{\beta\psi}(\theta) & B_{\beta\phi}(\theta) \\ B_{\psi\beta}(\theta) & B_\psi(\theta) & B_{\psi\phi}(\theta) \\ B_{\phi\beta}(\theta) & B_{\phi\psi}(\theta) & B_\phi(\theta) \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \mathcal{H}_\beta & \mathcal{H}_{\beta\psi} & \mathcal{H}_{\beta\phi} \\ \mathcal{H}_{\psi\beta} & \mathcal{H}_\psi & \mathcal{H}_{\psi\phi} \\ \mathcal{H}_{\phi\beta} & \mathcal{H}_{\phi\psi} & \mathcal{H}_\phi \end{pmatrix}. \tag{4.1}$$

With the notation introduced, the standard LM test statistic for $H_0 : r(\theta_0) = 0$ is defined in the following way (Newey and West 1987):

$$LM = N\, C'\left(\widehat{\theta}_{nT,r}\right) B^{-1}\left(\widehat{\theta}_{nT,r}\right) C\left(\widehat{\theta}_{nT,r}\right). \tag{4.2}$$

A similar test is the $C(\alpha)$ test.[5] This test is designed to deal with the nuisance parameters when testing the parameter of main interest (Bera and Bilias 2001). Lee and Yu (2012b) investigate the finite sample properties of this test for a cross-sectional autoregressive model. Their simulation results indicate that this test can be useful to test the possible presence of spatial correlation through a spatial lag in the spatial autoregressive (SAR) model. Here, we provide a general description of this test within the context of our SDPD model. By the implicit function theorem, the set of k_r restrictions on θ_0 can also be stated as $h(\xi_0) = \theta_0$, where $h : \mathbb{R}^{\bar{q}} \to \mathbb{R}^{2p+k_x+1}$ is continuously differentiable, ξ_0 contains the free parameters, and $\bar{q} = 2p + k_x + 1 - k_r$. Define $\widehat{\xi}_{nT} = \text{argmin}_\phi\, g'_{nT}(h(\xi))\,\widehat{\Sigma}^{-1}_{nT} g_{nT}(h(\xi))$. Then, we have $\widehat{\theta}_{nT,r} = h\left(\widehat{\xi}_{nT}\right)$. Let $\tilde{\xi}_{nT}$ be a consistent estimate of ξ_0. Denote $G_\xi(\theta) = \frac{1}{N}\frac{\partial g_{nT}(\theta)}{\partial \xi'}$, $C_\xi(\theta) = G'_\xi(\theta)\,\widehat{\Sigma}^{-1}_{nT}\bar{g}_{nT}(\theta)$, and $B_\xi(\theta) = G'_\xi(\theta)\,\widehat{\Sigma}^{-1}_{nT} G_\xi(\theta)$. Following the formulation suggested by Breusch and Pagan (1980), we state the $C(\alpha)$ test statistic in the following way

$$C(\alpha) = N\left[C'\left(h(\tilde{\xi}_{nT})\right)B^{-1}\left(h(\tilde{\xi}_{nT})\right)C\left(h(\tilde{\phi}_{nT})\right) - C'_\xi\left(h(\tilde{\xi}_{nT})\right)B^{-1}_\xi\left(h(\tilde{\xi}_{nT})\right)C_\xi\left(h(\tilde{\xi}_{nT})\right)\right]. \tag{4.3}$$

In (4.3), it is important to note that $\tilde{\xi}_{nT}$ can be any consistent estimator. In the case where $\tilde{\xi}_{nT}$ is an optimal GMME, the $C(\alpha)$ statistic reduces to LM statistic, since $C_\xi\left(h(\tilde{\xi}_{nT})\right) = 0$ by definition.[6] The asymptotic distributions of $C(\alpha)$ and LM are given in the following proposition.

[5] Breusch and Pagan (1980) call this test the pseudo-LM test, since its test statistic is very similar to the form of the LM statistic.

[6] In the context of ML estimation, the $C(\alpha)$ statistic reduces to the LM statistic when the restricted MLE is used. For details, see Bera and Bilias (2001).

Proposition 1. — Given our stated assumptions, we have the following results under $H_0 : r(\theta_0) = 0$:

$$LM \xrightarrow{d} \chi^2_{k_r}, \quad \text{and} \quad C(\alpha) \xrightarrow{d} \chi^2_{k_r}. \tag{4.4}$$

Proof. See Section C.1. □

Next, we consider the following joint null hypothesis:

$$H_0 : \lambda_0 = 0,\ \rho_0 = 0,\ \gamma_0 = 0, \quad H_A : \text{At least one parameter is not equal to zero.} \tag{4.5}$$

Under the joint null hypothesis, the model reduces to a two-way non-spatial panel data model which can be estimated by an OLSE (for the estimation of two-way models, see Baltagi (2008) and Hsiao (2014)). The joint null hypothesis can be tested either by LM or $C(\alpha)$. Let $\tilde{\theta}_{nT}$ be a constrained optimal GMME under the joint null hypothesis, and let $\widehat{\theta}_{nT}$ be any other consistent estimator of θ_0 under the null hypothesis. As stated in Newey and West (1987), the LM test statistic should be formulated with the optimal constrained GMME. Let $\vartheta = \left(\lambda^{'}, \rho^{'}, \gamma\right)^{'}$. Then, the LM test statistic for the joint null hypothesis can be expressed as

$$LM_J\big(\tilde{\theta}_{nT}\big) = N\, C_J^{'}\big(\tilde{\theta}_{nT}\big)\big[B_{\vartheta\cdot\beta}\left(\tilde{\theta}_{nT}\right)\big]^{-1} C_J\big(\tilde{\theta}_{nT}\big), \tag{4.6}$$

where $C_J^{'}\big(\tilde{\theta}_{nT}\big) = \big(C_\lambda^{'}\big(\tilde{\theta}_{nT}\big), C_\rho^{'}\big(\tilde{\theta}_{nT}\big), C_\gamma^{'}\big(\tilde{\theta}_{nT}\big)\big)^{'}$, $B_{\vartheta\cdot\beta}\big(\tilde{\theta}_{nT}\big) = B_\vartheta\big(\tilde{\theta}_{nT}\big) - B_{\vartheta\beta}\big(\tilde{\theta}_{nT}\big)B_\beta^{-1}\big(\tilde{\theta}_{nT}\big)B_{\beta\vartheta}\big(\tilde{\theta}_{nT}\big)$, $B_{\vartheta\beta}\big(\tilde{\theta}_{nT}\big) = B_{\beta\vartheta}^{'}\big(\tilde{\theta}_{nT}\big) = \big(B_{\lambda\beta}^{'}\big(\tilde{\theta}_{nT}\big), B_{\rho\beta}^{'}\big(\tilde{\theta}_{nT}\big), B_{\gamma\beta}^{'}\big(\tilde{\theta}_{nT}\big)\big)^{'}$, and

$$B_\vartheta\big(\tilde{\theta}_{nT}\big) = \begin{pmatrix} B_\lambda\big(\tilde{\theta}_{nT}\big) & B_{\lambda\rho}\big(\tilde{\theta}_{nT}\big) & B_{\lambda\gamma}\big(\tilde{\theta}_{nT}\big) \\ B_{\rho\lambda}\big(\tilde{\theta}_{nT}\big) & B_\rho\big(\tilde{\theta}_{nT}\big) & B_{\rho\gamma}\big(\tilde{\theta}_{nT}\big) \\ B_{\gamma\lambda}\big(\tilde{\theta}_{nT}\big) & B_{\gamma\rho}\big(\tilde{\theta}_{nT}\big) & B_\gamma\big(\tilde{\theta}_{nT}\big) \end{pmatrix}. \tag{4.7}$$

Similarly, the consistent estimator $\widehat{\theta}_{nT}$ can be used to formulate the following $C(\alpha)$ test for the joint null hypothesis:

$$C_J(\alpha) = N\left[C^{'}\big(\widehat{\theta}_{nT}\big)B^{-1}\big(\widehat{\theta}_{nT}\big)C\big(\widehat{\theta}_{nT}\big) - C_\beta^{'}\big(\widehat{\theta}_{nT}\big)B_\beta^{-1}\big(\widehat{\theta}_{nT}\big)C_\beta\big(\widehat{\theta}_{nT}\big)\right]. \tag{4.8}$$

The properties of the LM test can be investigated under a sequence of local alternatives (Bera and Bilias 2001; Bera and Yoon 1993; Bera et al. 2010; Davidson and MacKinnon 1987; Saikkonen 1989). Bera and Yoon (1993) and Bera et al. (2010) suggest robust LM tests when the alternative model is misspecified. We consider similar robust LM tests within the context of our model. In order to give a general result, we consider the LM test for $H_0^\psi : \psi_0 = 0$ when $H_0^\phi : \phi_0 = 0$, which can be stated as

$$LM_\psi = N\, C_\psi^{'}\big(\tilde{\theta}_{nT}\big)\big[B_{\psi\cdot\beta}\big(\tilde{\theta}_{nT}\big)\big]^{-1} C_\psi\big(\tilde{\theta}_{nT}\big), \tag{4.9}$$

where $B_{\psi\cdot\beta}\big(\tilde{\theta}_{nT}\big) = B_\psi\big(\tilde{\theta}_{nT}\big) - B_{\psi\beta}\big(\tilde{\theta}_{nT}\big)B_\beta^{-1}\big(\tilde{\theta}_{nT}\big)B_{\beta\psi}\big(\tilde{\theta}_{nT}\big)$. We investigate the asymptotic distribution of LM_ψ under the sequences of local alternatives $H_A^\psi : \psi = \psi_0 + \delta_\psi/\sqrt{N}$, and $H_A^\phi : \phi = \phi_0 + \delta_\phi/\sqrt{N}$, where $\left(\psi_0^{'}, \phi_0^{'}\right)^{'}$ is the vector of hypothesized values under the null, and δ_ψ and δ_ϕ are bounded vectors. The distribution of (4.9), under H_A^ψ and H_A^ϕ, can be investigated from the first order Taylor expansions of pseudo-scores $C_\psi\big(\tilde{\theta}_{nT}\big)$ and $C_\beta\big(\tilde{\theta}_{nT}\big)$ around

$\theta^* = \left(\beta_0^{'},\, \psi_0^{'} + \delta_\psi^{'}/\sqrt{N},\, \phi_0^{'} + \delta_\phi^{'}/\sqrt{N}\right)^{'}$. These expansions can be written as

$$\begin{aligned}
\sqrt{N}\, C_\psi\big(\tilde{\theta}_{nT}\big) &= \sqrt{N}\, C_\psi\big(\theta^*\big) - G_\psi^{'}\left(\theta^*\right) \widehat{\Sigma}_{nT}^{-1} G_\psi\left(\bar{\theta}\right)\delta_\psi - G_\psi^{'}\left(\theta^*\right) \widehat{\Sigma}_{nT}^{-1} G_\phi\left(\bar{\theta}\right)\delta_\phi \qquad (4.10)\\
&\quad + \sqrt{N}\, G_\psi^{'}\left(\theta^*\right) \widehat{\Sigma}_{nT}^{-1} G_\beta\left(\bar{\theta}\right)\left(\tilde{\beta}_{nT} - \beta_0\right) + o_p(1),\\
\sqrt{N}\, C_\beta\big(\tilde{\theta}_{nT}\big) &= \sqrt{N}\, C_\beta\left(\theta^*\right) - G_\beta^{'}\left(\theta^*\right) \widehat{\Sigma}_{nT}^{-1} G_\psi\left(\bar{\theta}\right)\delta_\psi - G_\beta^{'}\left(\theta^*\right) \widehat{\Sigma}_{nT}^{-1} G_\phi\left(\bar{\theta}\right)\delta_\phi \qquad (4.11)\\
&\quad + \sqrt{N}\, G_\beta^{'}\left(\theta^*\right) \widehat{\Sigma}_{nT}^{-1} G_\beta\left(\bar{\theta}\right)\left(\tilde{\beta}_{nT} - \beta_0\right) + o_p(1),
\end{aligned}$$

where $\bar{\theta}$ lies between $\tilde{\theta}_{nT}$ and θ^*. Note that $\theta^* = \theta_0 + o_p(1)$ implies $\bar{\theta} = \theta_0 + o_p(1)$. By Lemma 1, we have $B(\theta^*) = \mathcal{H} + o_p(1)$, and $G^{'}(\theta^*)\widehat{\Sigma}_{nT} = \mathcal{G}^{'}\Sigma_{nT} + o_p(1)$. Then, from (4.10) and (4.11), we get the following fundamental result:

$$\begin{aligned}
\sqrt{N}\, C_\psi\big(\tilde{\theta}_{nT}\big) &= \left[\mathcal{G}_\psi^{'}\Sigma_{nT}^{-1} - \mathcal{H}_{\psi\beta}\mathcal{H}_\beta^{-1}\mathcal{G}_\beta^{'}\Sigma_{nT}^{-1}\right]\frac{1}{\sqrt{N}}\, g_{nT}(\theta_0) \qquad (4.12)\\
&\quad - \left[\mathcal{H}_\psi - \mathcal{H}_{\psi\beta}\mathcal{H}_\beta^{-1}\mathcal{H}_{\beta\psi}\right]\delta_\psi - \left[\mathcal{H}_{\psi\phi} - \mathcal{H}_{\psi\beta}\mathcal{H}_\beta^{-1}\mathcal{H}_{\beta\phi}\right]\delta_\phi + o_p(1).
\end{aligned}$$

By Lemma 1, we have $\frac{1}{\sqrt{N}} g_{nT}(\theta_0) \xrightarrow{d} N\big(0, \operatorname{plim}_{n\to\infty}\Sigma_{nT}\big)$, and thus (4.12) implies that $\sqrt{N}\, C_\psi\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} N\big(-\mathcal{H}_{\psi\cdot\beta}\delta_\psi - \mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi, \mathcal{H}_{\psi\cdot\beta}\big)$, where $\mathcal{H}_{\psi\cdot\beta} = \left[\mathcal{H}_\psi - \mathcal{H}_{\psi\beta}\mathcal{H}_\beta^{-1}\mathcal{H}_{\beta\psi}\right]$, and $\mathcal{H}_{\psi\phi\cdot\beta} = \left[\mathcal{H}_{\psi\phi} - \mathcal{H}_{\psi\beta}\mathcal{H}_\beta^{-1}\mathcal{H}_{\beta\phi}\right]$. Hence, $LM_\psi\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_{k_\psi}^2(\vartheta_1)$ under H_A^ψ and H_A^ϕ, where $\vartheta_1 = \delta_\psi^{'}\mathcal{H}_{\psi\cdot\beta}\delta_\psi + \delta_\psi^{'}\mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi + \delta_\phi^{'}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\delta_\psi + \delta_\phi^{'}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\mathcal{H}_{\psi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi$ is the non-centrality parameter.[7] We provide the distributional results for $LM_\psi\big(\tilde{\theta}_{nT}\big)$ and its robust version in the following proposition.

Proposition 2. — Given our stated assumptions, the following results hold.

1. Under H_A^ψ and H_A^ϕ, we have

$$LM_\psi\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_{k_\psi}^2\left(\vartheta_1\right), \qquad (4.13)$$

where $\vartheta_1 = \delta_\psi^{'}\mathcal{H}_{\psi\cdot\beta}\delta_\psi + \delta_\psi^{'}\mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi + \delta_\phi^{'}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\delta_\psi + \delta_\phi^{'}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\mathcal{H}_{\psi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi$.

2. Under H_A^ψ and H_0^ϕ, we have

$$LM_\psi\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_{k_\psi}^2\left(\vartheta_2\right), \qquad (4.14)$$

where $\vartheta_2 = \delta_\psi^{'}\mathcal{H}_{\psi\cdot\beta}\delta_\psi$.

3. Under H_0^ψ and H_A^ϕ, we have

$$LM_\psi\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_{k_\psi}^2\left(\vartheta_3\right), \qquad (4.15)$$

where $\vartheta_3 = \delta_\phi^{'}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\mathcal{H}_{\psi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi$.

4. Let $C_\psi^{\star'}\big(\tilde{\theta}_{nT}\big) = \left[C_\psi\big(\tilde{\theta}_{nT}\big) - B_{\psi\phi\cdot\beta}\big(\tilde{\theta}_{nT}\big)B_{\phi\cdot\beta}^{-1}\big(\tilde{\theta}_{nT}\big)C_\phi\big(\tilde{\theta}_{nT}\big)\right]$ be the adjusted pseudo-score, where $B_{\psi\phi\cdot\beta}\big(\tilde{\theta}_{nT}\big) = B_{\psi\phi}\big(\tilde{\theta}_{nT}\big) - B_{\psi\beta}\big(\tilde{\theta}_{nT}\big)B_\beta^{-1}\big(\tilde{\theta}_{nT}\big)B_{\beta\phi}\big(\tilde{\theta}_{nT}\big)$, and $B_{\phi\cdot\beta}\big(\tilde{\theta}_{nT}\big) = B_\phi\big(\tilde{\theta}_{nT}\big) -$

[7] For the definition of non-centrality chi-square distribution, see Anderson (2003, p.81-82).

$B_{\phi\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\phi}(\tilde{\theta}_{nT})$. Under H_0^{ψ} and irrespective of whether H_0^{ϕ} or H_A^{ϕ} holds, we have

$$LM_{\psi}^{\star}(\tilde{\theta}_{nT}) = N\, C_{\psi}^{\star\prime}(\tilde{\theta}_{nT})\big[B_{\psi\cdot\beta}(\tilde{\theta}_{nT}) - B_{\psi\phi\cdot\beta}(\tilde{\theta}_{nT})B_{\phi\cdot\beta}^{-1}(\tilde{\theta}_{nT})B_{\psi\phi\cdot\beta}^{\prime}(\tilde{\theta}_{nT})\big]^{-1}C_{\psi}^{\star}(\tilde{\theta}_{nT}) \xrightarrow{d} \chi^2_{k_{\psi}}. \tag{4.16}$$

5. Under H_A^{ψ} and H_0^{ϕ}, we have

$$LM_{\psi}^{\star}(\tilde{\theta}_{nT}) \xrightarrow{d} \chi^2_{k_{\psi}}(\vartheta_4), \tag{4.17}$$

where $\vartheta_4 = \delta_{\psi}^{\prime}(\mathcal{H}_{\psi\cdot\beta} - \mathcal{H}_{\psi\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}^{\prime})\delta_{\psi}$.

Proof. See Section C.2. □

There are three important observations regarding to the results presented in Proposition 2. First, the one directional test has a non-central chi-square distribution when the alternative model is misspecified, i.e., when the alternative model includes ϕ_0. The non-centrality parameter is $\vartheta_3 = \delta_{\phi}^{\prime}\mathcal{H}_{\psi\phi\cdot\beta}^{\prime}\mathcal{H}_{\psi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}\delta_{\phi}$, which would be zero if and only if $\mathcal{H}_{\psi\phi\cdot\beta} = 0$. Second, the robust test $LM_{\psi}^{\star}(\tilde{\theta}_{nT})$ has a central chi-square distribution even when the alternative model is locally misspecified. Finally, $LM_{\psi}^{\star}(\tilde{\theta}_{nT})$ has less asymptotic power than $LM_{\psi}(\tilde{\theta}_{nT})$, since $\vartheta_2 - \vartheta_4 \geq 0$ under H_A^{ψ} and H_0^{ϕ}.

Proposition 2 provides a template that can be used to determine the test statistics for the following hypotheses:

1. The null hypothesis for the contemporaneous spatial lag terms: $H_0^{\lambda}: \lambda_0 = 0$ in the presence of ρ_0 and γ_0.
2. The null hypothesis for the spatial lag terms at time $t-1$: $H_0^{\rho}: \rho_0 = 0$ in the presence of λ_0 and γ_0.
3. The null hypothesis for the time lag term: $H_0^{\gamma}: \gamma_0 = 0$ in the presence of λ_0 and ρ_0.

In the following, we provide the test statistic for each hypothesis and leave the detailed derivations to Appendix B. We start with $H_0^{\lambda}: \lambda_0 = 0$. In the context of this hypothesis, $\phi = (\rho^{\prime}, \gamma)^{\prime}$. Then, the one directional test can be written as

$$LM_{\lambda}(\tilde{\theta}_{nT}) = N\, C_{\lambda}^{\prime}(\tilde{\theta}_{nT})\big[B_{\lambda\cdot\beta}(\tilde{\theta}_{nT})\big]^{-1}C_{\lambda}(\tilde{\theta}_{nT}), \tag{4.18}$$

where $B_{\lambda\cdot\beta}(\tilde{\theta}_{nT}) = B_{\lambda}(\tilde{\theta}_{nT}) - B_{\lambda\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\lambda}(\tilde{\theta}_{nT})$. Then, $LM_{\lambda}(\tilde{\theta}_{nT}) \xrightarrow{d} \chi^2_p(\vartheta_2)$ under H_A^{λ} and H_0^{ϕ}; and $LM_{\lambda}(\tilde{\theta}_{nT}) \xrightarrow{d} \chi^2_p(\vartheta_3)$ under H_0^{λ} and H_A^{ϕ}, where $\vartheta_2 = \delta_{\lambda}^{\prime}\mathcal{H}_{\lambda\cdot\beta}\delta_{\lambda}$ and $\vartheta_3 = \delta_{\phi}^{\prime}\mathcal{H}_{\lambda\phi\cdot\beta}^{\prime}\mathcal{H}_{\lambda\cdot\beta}^{-1}\mathcal{H}_{\lambda\phi\cdot\beta}\delta_{\phi}$. The robust version is stated as

$$LM_{\lambda}^{\star}(\tilde{\theta}_{nT}) = N\, C_{\lambda}^{\star\prime}(\tilde{\theta}_{nT})\big[B_{\lambda\cdot\beta}(\tilde{\theta}_{nT}) - B_{\lambda\phi\cdot\beta}(\tilde{\theta}_{nT})B_{\phi\cdot\beta}^{-1}(\tilde{\theta}_{nT})B_{\lambda\phi\cdot\beta}^{\prime}(\tilde{\theta}_{nT})\big]^{-1}C_{\lambda}^{\star}(\tilde{\theta}_{nT}), \tag{4.19}$$

where $C_{\lambda}^{\star}(\tilde{\theta}_{nT}) = \big[C_{\lambda}(\tilde{\theta}_{nT}) - B_{\lambda\phi\cdot\beta}(\tilde{\theta}_{nT})B_{\phi\cdot\beta}^{-1}(\tilde{\theta}_{nT})C_{\phi}(\tilde{\theta}_{nT})\big]$ is the adjusted score. Irrespective of whether H_0^{ϕ} or H_A^{ϕ} holds, $LM_{\lambda}^{\star}(\tilde{\theta}_{nT})$ has an asymptotic χ^2_p distribution under H_0^{λ} by Proposition 2. Finally, under H_A^{λ} and H_0^{ϕ}, we have $LM_{\lambda}^{\star}(\tilde{\theta}_{nT}) \xrightarrow{d} \chi^2_p(\vartheta_4)$, where $\vartheta_4 = \delta_{\lambda}^{\prime}(\mathcal{H}_{\lambda\cdot\beta} - \mathcal{H}_{\lambda\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\lambda\phi\cdot\beta}^{\prime})\delta_{\lambda}$.

Next, we consider $H_0^\rho : \rho_0 = 0$. In the context of this hypothesis, $\phi = \big(\lambda', \gamma\big)'$. The one directional test can be written as

$$LM_\rho\big(\tilde{\theta}_{nT}\big) = N\, C_\rho'\big(\tilde{\theta}_{nT}\big)\big[B_{\rho\cdot\beta}\big(\tilde{\theta}_{nT}\big)\big]^{-1} C_\rho\big(\tilde{\theta}_{nT}\big), \tag{4.20}$$

where $B_{\rho\cdot\beta}\big(\tilde{\theta}_{nT}\big) = B_\rho\big(\tilde{\theta}_{nT}\big) - B_{\rho\beta}\big(\tilde{\theta}_{nT}\big)B_\beta^{-1}\big(\tilde{\theta}_{nT}\big)B_{\beta\rho}\big(\tilde{\theta}_{nT}\big)$. Proposition 2 implies that $LM_\rho\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_p^2\left(\vartheta_2\right)$ under H_A^ρ and H_0^ϕ; and $LM_\rho\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_p^2\left(\vartheta_3\right)$ under H_0^ρ and H_A^ϕ, where $\vartheta_2 = \delta_\rho' \mathcal{H}_{\rho\cdot\beta}\delta_\rho$ and $\vartheta_3 = \delta_\phi' \mathcal{H}_{\rho\phi\cdot\beta}' \mathcal{H}_{\rho\cdot\beta}^{-1} \mathcal{H}_{\rho\phi\cdot\beta}\delta_\phi$. The robust version of $LM_\rho\big(\tilde{\theta}_{nT}\big)$ is stated as

$$LM_\rho^\star\big(\tilde{\theta}_{nT}\big) = N\, C_\rho^{\star\prime}\big(\tilde{\theta}_{nT}\big)\big[B_{\rho\cdot\beta}\big(\tilde{\theta}_{nT}\big) - B_{\rho\phi\cdot\beta}\big(\tilde{\theta}_{nT}\big)B_{\phi\cdot\beta}^{-1}\big(\tilde{\theta}_{nT}\big)B_{\rho\phi\cdot\beta}'\big(\tilde{\theta}_{nT}\big)\big]^{-1} C_\rho^\star\big(\tilde{\theta}_{nT}\big), \tag{4.21}$$

where $C_\rho^\star\big(\tilde{\theta}_{nT}\big) = \big[C_\rho\big(\tilde{\theta}_{nT}\big) - B_{\rho\phi\cdot\beta}\big(\tilde{\theta}_{nT}\big)B_{\phi\cdot\beta}^{-1}\big(\tilde{\theta}_{nT}\big)C_\phi\big(\tilde{\theta}_{nT}\big)\big]$. The asymptotic null distribution of $LM_\rho^\star\big(\tilde{\theta}_{nT}\big)$ is χ_p^2, irrespective of whether H_0^ϕ or H_A^ϕ holds. Finally, under H_A^ρ and H_0^ϕ, we have $LM_\rho^\star\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_p^2\left(\vartheta_4\right)$, where $\vartheta_4 = \delta_\rho'\big(\mathcal{H}_{\rho\cdot\beta} - \mathcal{H}_{\rho\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\rho\phi\cdot\beta}'\big)\delta_\rho$.

Finally, we consider $H_0^\gamma : \gamma_0 = 0$. Here, we have $\phi = \big(\lambda', \rho'\big)'$. The one directional test can be written as

$$LM_\gamma\big(\tilde{\theta}_{nT}\big) = N\, C_\gamma'\big(\tilde{\theta}_{nT}\big)\big[B_{\gamma\cdot\beta}\big(\tilde{\theta}_{nT}\big)\big]^{-1} C_\gamma\big(\tilde{\theta}_{nT}\big), \tag{4.22}$$

where $B_{\gamma\cdot\beta}\big(\tilde{\theta}_{nT}\big) = B_\gamma\big(\tilde{\theta}_{nT}\big) - B_{\gamma\beta}\big(\tilde{\theta}_{nT}\big)B_\beta^{-1}\big(\tilde{\theta}_{nT}\big)B_{\beta\gamma}\big(\tilde{\theta}_{nT}\big)$. Then, $LM_\gamma\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_1^2\left(\vartheta_2\right)$ under H_A^γ and H_0^ϕ; and $LM_\gamma\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_1^2\left(\vartheta_3\right)$ under H_0^γ and H_A^ϕ, where $\vartheta_2 = \delta_\gamma' \mathcal{H}_{\gamma\cdot\beta}\delta_\gamma$ and $\vartheta_3 = \delta_\phi' \mathcal{H}_{\gamma\phi\cdot\beta}' \mathcal{H}_{\gamma\cdot\beta}^{-1} \mathcal{H}_{\gamma\phi\cdot\beta}\delta_\phi$. The robust version is stated as

$$LM_\gamma^\star\big(\tilde{\theta}_{nT}\big) = N\, C_\gamma^{\star\prime}\big(\tilde{\theta}_{nT}\big)\big[B_{\gamma\cdot\beta}\big(\tilde{\theta}_{nT}\big) - B_{\gamma\phi\cdot\beta}\big(\tilde{\theta}_{nT}\big)B_{\phi\cdot\beta}^{-1}\big(\tilde{\theta}_{nT}\big)B_{\gamma\phi\cdot\beta}'\big(\tilde{\theta}_{nT}\big)\big]^{-1} C_\gamma^\star\big(\tilde{\theta}_{nT}\big), \tag{4.23}$$

where $C_\gamma^\star\big(\tilde{\theta}_{nT}\big) = \big[C_\gamma\big(\tilde{\theta}_{nT}\big) - B_{\gamma\phi\cdot\beta}\big(\tilde{\theta}_{nT}\big)B_{\phi\cdot\beta}^{-1}\big(\tilde{\theta}_{nT}\big)C_\phi\big(\tilde{\theta}_{nT}\big)\big]$. The asymptotic null distribution of $LM_\gamma^\star\big(\tilde{\theta}_{nT}\big)$ is χ_1^2, irrespective of whether H_0^ϕ or H_A^ϕ holds. Finally, under H_A^γ and H_0^ϕ, we have $LM_\gamma^\star\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_1^2\left(\vartheta_4\right)$, where $\vartheta_4 = \delta_\gamma'\big(\mathcal{H}_{\gamma\cdot\beta} - \mathcal{H}_{\gamma\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\gamma\phi\cdot\beta}'\big)\delta_\gamma$.

5 Monte Carlo Simulation

In this section, we describe the details of Monte Carlo design for our analysis. Our design is based on Lee and Yu (2014) and Yang (2015). For the model in (2.1), we will focus on the case where $p = 1$:

$$Y_{nt} = \lambda_0 W_n Y_{nt} + \gamma_0 Y_{n,t-1} + \rho_0 W_n Y_{n,t-1} + X_{nt}\beta_0 + \mathbf{c}_{n0} + \alpha_{t0} l_n + V_{nt}, \tag{5.1}$$

for $t = 1, 2, \ldots, T$. We generate the weights matrix according to (i) Rook contiguity and (ii) Queen contiguity. The n spatial units are randomly permuted and allocated into a lattice of $k \times m$ squares, where $m \geq n$. In the Rook contiguity, $w_{ij,n} = 1$ if the spatial unit j is in a square that is adjacent (left/right/above or below) to the square of the spatial unit i. In the Queen contiguity, $w_{ij,n} = 1$ if the spatial unit j is in a square that is adjacent to or shares a corner with the square of the spatial unit i. In both cases, W_n is row normalized.

We allow for two exogenous regressors. The first one is generated as $X_{1,nt} = \Psi_n + 0.01\, t\, l_n + U_{nt}$, where $U_{nt} = 0.5\, U_{n,t-1} + \varepsilon_{nt} + 0.5\, \varepsilon_{n,t-1}$ and $\varepsilon_{nt} \sim N(0_{n\times 1}, 2I_n)$. Furthermore, $\Psi_n = \Upsilon_n + 1/(T+m+$

1) $\sum_{t=-m}^{T}\varepsilon_{nt}$, where $\Upsilon_n \sim N(0_{n\times 1}, I_n)$ and $m=20$. Then, $X_{nt} = \big(X_{1,nt}, W_n X_{2,nt}\big)$ where $X_{2,nt} \sim N(0_{n\times 1}, I_n)$. We set $\beta_0 = (1.2, 0.6)$. For the individual effects, we let $\mathbf{c}_{n0} = (1/T)\sum_{t=1}^{T} X_{1,nt}$, and draw α_{t0} from $N(0,1)$. For the error term $V_{i,nt}$, we specify two cases: (i) $V_{i,nt} \sim N(0,1)$ and (ii) $V_{i,nt} \sim \text{Gamma}(1,1) - 1$. The data generating process has $21+T$ periods and the last $T+1$ periods are used for estimation. For the sample size, we use the following n and T combinations: $(n,\, T) = \{(100, 10),\, (20, 200)\}$.[8]

Under the null model (i.e., $\lambda_0 = \gamma_0 = \rho_0 = 0$), (5.1) reduces to a two-way error model (2WE). We can employ seven different specifications for the alternative model. We choose to focus on the following four specifications as they are more common in empirical applications. The first specification is a dynamic panel data model with no spatial effects (DPD), i.e., when $\lambda_0 = \rho_0 = 0$ and $\gamma_0 \neq 0$ in (5.1). The second specification is a spatial static panel model (SSPD), i.e., when $\lambda_0 \neq 0$ and $\rho_0 = \gamma_0 = 0$ in (5.1). The third specification is a spatial dynamic panel data model with no spatial-time lag (SDPDW), i.e., when $\rho_0 = 0$, $\lambda_0 \neq 0$ and $\gamma_0 \neq 0$ in (5.1). The final specification for the alternative modes is the spatial dynamic panel data model (SDPD), i.e., when $\rho_0 \neq 0$, $\lambda_0 \neq 0$ and $\gamma_0 \neq 0$ in (5.1). Note that the first three alternative models can be considered as the null models for the one-directional tests and their robust counterparts in the following way: (i) the DPD model for LM_ρ, $\text{LM}^\star_\rho$, LM_λ and $\text{LM}^\star_\lambda$; (ii) the SSP model for LM_ρ, $\text{LM}^\star_\rho$, LM_γ and $\text{LM}^\star_\gamma$; (iii) the SDPDW model for LM_ρ and $\text{LM}^\star_\rho$. We let λ_0, γ_0 and ρ_0 take values from $\{-0.3, -0.1, , -0.05, 0.05, 0.1, 0.3\}$ for the alternative models. Hence, the DPD, SSPD, SDPDW and SDPD specifications yield respectively 6, 6, 16 and 216 combinations. Resampling is carried out for $5,000$ times.

Table 1 summarizes the null hypotheses and the respective test statistics along with the source of misspecification in each hypothesis considered in the Monte Carlo study. For example, the source of misspecification for $\text{H}_0 : \lambda_0 = 0$ is the presence of ρ_0 and γ_0 in the alternative model. All test statistics presented in Table 1 are computed by the estimates from the 2WE model. For the test statistics, we also need to specify the set of moment functions. The set of linear moments consists of $Q_{nt} = \big(Y_{n,t-1},\, W_n Y_{n,t-1},\, W_n^2 Y_{n,t-1},\, X^*_{n,t},\, W_n X^*_{n,t},\, W_n^2 X^*_{n,t}\big)$ for $t = 1, 2, \ldots, T-1$. For the quadratic moments, we employ $P_{n1} = W_n - \text{tr}(W_n J_n)/(n-1) J_n$ and $P_{n2} = W_n^2 - \text{tr}(W_n^2 J_n)/(n-1)J_n$. Note that we do not consider the conditional tests that require a restricted GMME (see Proposition 1) for the computation of the test statistics. Here our aim is to compare the performance of the robust tests with their non-robust counterparts once the estimates of the simple 2WE model are available.

5.1 Results on Size Properties

A P value discrepancy plots is generated from the empirical distribution function (edf) of p values. To see how, let τ denote a test statistic, and τ_j for $j = 1, \ldots, \mathcal{R}$ be the $\mathcal{R}$ realizations of τ generated in a Monte Carlo experiment. Let $F(x)$ denote the cumulative distribution function (cdf) of the asymptotic distribution of τ evaluated at the level x. Then, the p value associated with τ_j, denoted by $p(\tau_j)$, is given by $p(\tau_j) = 1 - F(\tau_j)$. An estimate of the cdf of $p(\tau)$ can be constructed simply from the edf of $p(\tau_j)$. Consider a sequence of levels denoted by $\{x_i\}$ for $i = 1, \ldots, m$ from the interval (0, 1). Then, an estimate of the cdf of $p(\tau)$ is given by $\widehat{F}(x_i) = \sum_{j=1}^{\mathcal{R}} \mathbf{1}\big(p(\tau_j) \leq x_i\big)/\mathcal{R}$.[9]

The P value discrepancy plot is created by plotting $\widehat{F}(x_i) - x_i$ against x_i under the assumption that the true data generating process is characterized by the null hypothesis. To asses the

[8]For the sake of brevity, we only provide estimation results for $(n,\, T) = (100, 10)$.

[9]We choose the following sequence and focus on the levels smaller than or equal to 0.1: $\{x_i\}_{i=1}^{m} = \{0.001 : 0.001 : 0.010 \quad 0.015 : 0.005 : 0.990 \quad 0.991 : 0.001 : 0.999\}$.

Table 1: Summary of test statistics

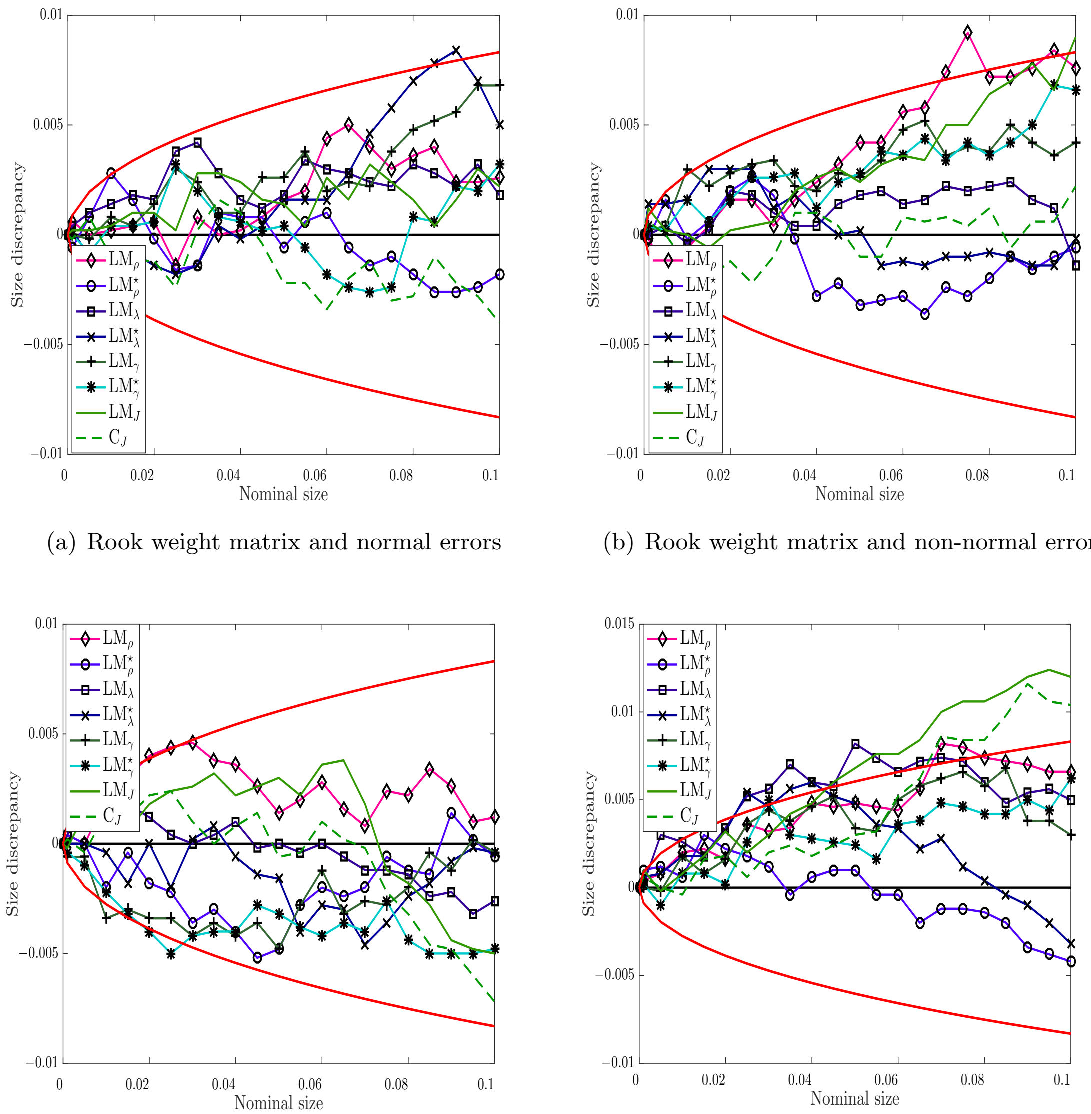
Null hypothesis	Parameter		Test statistic
	Spatial time lag: ρ_0	Time lag: γ_0	
$H_0 : \lambda_0 = 0$	Set to zero	Set to zero	LM_λ in (4.18)
$H_0 : \lambda_0 = 0$	Unrestricted, not estimated	Unrestricted, not estimated	$LM_\lambda^\star$ in (4.19)
	Contemporaneous spatial lag: λ_0	Time lag: γ_0	
$H_0 : \rho_0 = 0$	Set to zero	Set to zero	LM_ρ in (4.20)
$H_0 : \rho_0 = 0$	Unrestricted, not estimated	Unrestricted, not estimated	$LM_\rho^\star$ in (4.21)
	Contemporaneous spatial lag: λ_0	Spatial time lag: ρ_0	
$H_0 : \gamma_0 = 0$	Set to zero	Set to zero	LM_γ in (4.22)
$H_0 : \gamma_0 = 0$	Unrestricted, not estimated	Unrestricted, not estimated	$LM_\gamma^\star$ in (4.23)
$H_0 : \lambda_0 = 0, \rho_0 = 0, \gamma_0 = 0$	–	–	LM_J in (4.6)
$H_0 : \lambda_0 = 0, \rho_0 = 0, \gamma_0 = 0$	–	–	C_J in (4.8)

significance of discrepancies in a P value discrepancy plot, we construct a point-wise 95% confidence interval for a nominal size by using a normal approximation to the binomial distribution (Anselin et al. 1996). Let α denote the nominal size at which the test is carried out. Using a normal approximation to the binomial distribution, a point-wise 95% confidence interval centered on α would be given by $\alpha \pm 1.96\,[\alpha(1-\alpha)/\mathcal{R}]^{1/2}$, and thus it would include rejection rates between $\alpha - 1.96\,[\alpha(1-\alpha)/\mathcal{R}]^{1/2}$ and $\alpha + 1.96\,[\alpha(1-\alpha)/\mathcal{R}]^{1/2}$. We use this approach to insert a 95% confidence interval in a P value discrepancy plot. In the discrepancy plots, the interval will be represented by the red solid lines.

Table 2: Empirical sizes when H_0: The DPD model and $(n, T) = (100, 10)$

	Normal Distribution				*Gamma Distribution*			
γ_0	LM_ρ	$LM_\rho^\star$	LM_λ	$LM_\lambda^\star$	LM_ρ	$LM_\rho^\star$	LM_λ	$LM_\lambda^\star$
				Rook				
-0.30	0.046	0.015	0.042	0.005	0.047	0.016	0.042	0.008
-0.10	0.044	0.038	0.042	0.039	0.040	0.042	0.041	0.037
-0.05	0.040	0.049	0.048	0.051	0.043	0.045	0.047	0.046
0.05	0.061	0.046	0.061	0.056	0.057	0.051	0.056	0.052
0.10	0.074	0.042	0.064	0.039	0.070	0.041	0.061	0.043
0.30	0.135	0.028	0.100	0.024	0.128	0.035	0.099	0.028
				Queen				
-0.30	0.063	0.020	0.053	0.012	0.062	0.018	0.049	0.011
-0.10	0.044	0.047	0.046	0.043	0.039	0.038	0.044	0.038
-0.05	0.049	0.053	0.051	0.048	0.044	0.048	0.042	0.044
0.05	0.055	0.046	0.058	0.051	0.062	0.049	0.055	0.050
0.10	0.075	0.050	0.060	0.050	0.070	0.045	0.061	0.043
0.30	0.099	0.012	0.062	0.017	0.083	0.015	0.051	0.020

To save space, the size results based on the 2WE model will be presented through the P value discrepancy plots whereas the size results based on the DPD, SSPD and SDPDW models will be summarized in tables. Note that in our design we allow for 6 different values for λ_0, γ_0 and ρ_0, which would yield 216 P value discrepancy plots for each. Hence, when the null model is one of the DPD, SSPD and SDPDW models, we focus solely on the nominal size of 5% and provide size



(a) Rook weight matrix and normal errors

(b) Rook weight matrix and non-normal errors

(c) Queen weight matrix and normal errors

(d) Queen weight matrix and non-normal errors

Figure 1: Size discrepancy plots when $(n, T) = (100, 10)$.

deviations at this level only. The general observations on the size properties of tests from Figure 1 and Tables 2 through 4 are listed as follows.

1. Fisgure 1 presents the size discrepancy plots when the null model is 2WE. The results show that all tests have little size distortions and their size discrepancies generally lie inside the 95% confidence interval. The size discrepancies are relatively larger in the case of queen weight matrix and non-normal errors.

2. Table 2 provide some evidences on the magnitude of size distortions as a function of the size of local misspecification of the alternative model, the DPD model. One would expect to see robust versions of the one directional tests, $\mathrm{LM}^\star_\rho$ and $\mathrm{LM}^\star_\lambda$, to perform better than LM_ρ and LM_λ, respectively, when the magnitude of misspecification is small. Overall, this seems to

be the case. For example, when the value of γ_0 is 0.05 in absolute value in the true model, the actual size of the robust tests are very close to the nominal size of 5%. However, as the misspecification deteriorates, this property of the robust tests vanish as expected.

3. Similar results hold for Table 3 as well, the robust versions of the one directional tests, $\mathrm{LM}_\rho^\star$ and $\mathrm{LM}_\gamma^\star$, perform better than LM_ρ and LM_γ, respectively, when λ_0 deviates locally from zero in the null model.

4. Tables 4 and 5 confirms our previous findings: $\mathrm{LM}_\rho^\star$ perform better than LM_ρ, when λ_0 and γ_0 deviate locally from zero. For example, in Table 4, when true values of λ_0 and γ_0 are 0.1, the actual size of $\mathrm{LM}_\rho^\star$ is 0.045 at the 5% level in the case of normal errors, whereas the actual size of LM_ρ is 0.985.

5. Recall that the robust tests use the residuals from the estimation of 2WE model and implements a correction on the test statistics for a local misspecification of the alternative model, i.e., ignoring the spatial component(s). The bias in these residuals depends on the strength of spatial dependence as well as on the connectedness of the weights matrix. Therefore, we can expect poor performance for the robust tests as spatial parameters deviate from zero substantially in the alternative model.

6. Finally, Tables 2, 4 and 5 indicate that as the temporal dependence strengthens, i.e., the misspecification in γ_0 gets larger in absolute value, the performance of the robust one-directional tests deteriorates significantly relative to their non-robust counterparts. This is not surprising in the sense that the bias in the residuals from the estimation of 2WE model increase as the dependence over time strengthens.

Table 3: Empirical sizes when H_0: The SSPD model and $(n, T) = (100, 10)$

	Normal Distribution				*Gamma Distribution*			
λ_0	LM_ρ	$\mathrm{LM}_\rho^\star$	LM_γ	$\mathrm{LM}_\gamma^\star$	LM_ρ	$\mathrm{LM}_\rho^\star$	LM_γ	$\mathrm{LM}_\gamma^\star$
				Rook				
-0.30	1.000	0.791	1.000	0.999	1.000	0.793	1.000	1.000
-0.10	0.913	0.051	0.334	0.184	0.913	0.048	0.335	0.168
-0.05	0.394	0.053	0.087	0.071	0.379	0.052	0.085	0.065
0.05	0.326	0.048	0.077	0.068	0.336	0.051	0.073	0.064
0.10	0.853	0.051	0.204	0.136	0.863	0.053	0.215	0.143
0.30	1.000	0.730	0.998	0.997	1.000	0.708	0.999	0.998
				Queen				
-0.30	0.994	0.134	0.604	0.431	0.997	0.144	0.614	0.451
-0.10	0.393	0.058	0.070	0.068	0.374	0.055	0.072	0.064
-0.05	0.134	0.052	0.056	0.057	0.132	0.047	0.049	0.049
0.05	0.171	0.046	0.073	0.063	0.187	0.045	0.060	0.054
0.10	0.550	0.053	0.103	0.071	0.539	0.055	0.116	0.073
0.30	0.999	0.202	0.972	0.970	0.999	0.195	0.972	0.969

5.2 Results on Power Properties

To investigate power properties of all tests, we use the approach described in Davidson and MacKinnon (1998) to generate the size power curves against the actual size obtained under the cor-

Table 4: Empirical sizes when H_0: The SDPDW model and $(n, T) = (100, 10)$: Rook

		Normal Distribution		*Gamma Distribution*	
λ_0	γ_0	LM_ρ	$LM^\star_\rho$	LM_ρ	$LM^\star_\rho$
-0.30	-0.30	0.580	0.299	0.578	0.310
-0.30	-0.10	0.999	0.833	1.000	0.831
-0.30	-0.05	1.000	0.831	1.000	0.830
-0.30	0.05	1.000	0.772	1.000	0.778
-0.30	0.10	1.000	0.885	1.000	0.892
-0.30	0.30	1.000	1.000	1.000	1.000
-0.10	-0.30	0.120	0.045	0.118	0.043
-0.10	-0.10	0.466	0.039	0.466	0.039
-0.10	-0.05	0.734	0.046	0.751	0.048
-0.10	0.05	0.971	0.048	0.974	0.042
-0.10	0.10	0.990	0.047	0.987	0.049
-0.10	0.30	1.000	0.739	0.999	0.739
-0.05	-0.30	0.073	0.025	0.067	0.024
-0.05	-0.10	0.128	0.044	0.137	0.045
-0.05	-0.05	0.242	0.050	0.255	0.047
-0.05	0.05	0.515	0.051	0.502	0.048
-0.05	0.10	0.612	0.049	0.610	0.046
-0.05	0.30	0.819	0.219	0.835	0.215
0.05	-0.30	0.068	0.018	0.062	0.015
0.05	-0.10	0.121	0.046	0.115	0.036
0.05	-0.05	0.207	0.049	0.208	0.053
0.05	0.05	0.474	0.051	0.469	0.053
0.05	0.10	0.557	0.042	0.585	0.045
0.05	0.30	0.598	0.022	0.597	0.017
0.10	-0.30	0.133	0.031	0.134	0.035
0.10	-0.10	0.360	0.042	0.347	0.051
0.10	-0.05	0.639	0.053	0.639	0.056
0.10	0.05	0.956	0.054	0.957	0.055
0.10	0.10	0.985	0.045	0.985	0.046
0.10	0.30	0.990	0.151	0.991	0.157
0.30	-0.30	0.763	0.296	0.764	0.302
0.30	-0.10	0.976	0.746	0.970	0.768
0.30	-0.05	1.000	0.757	1.000	0.754
0.30	0.05	1.000	0.643	1.000	0.652
0.30	0.10	1.000	0.637	1.000	0.627
0.30	0.30	1.000	1.000	1.000	1.000

responding null hypothesis. Therefore, two experiments need to be carried out. First, the data generating process under the alternative hypothesis is used to generate the edf of p-values. We denote the resulting edf by $\widetilde{F}(x)$. Second, the data generating process satisfies the null hypothesis, and as before $\widehat{F}(x)$ denotes the resulting edf of p-values. Then, a size-power curve is generated by plotting $\widetilde{F}(x_i)$ against $\widehat{F}(x_i)$ for $i = 1, \ldots, m$. As stated in Davidson and MacKinnon (1998), the size-power curve avoids the size adjustments made to generate the power curves.

Table 5: Empirical sizes when H_0: The SDPDW model and $(n, T) = (100, 10)$: Queen

		Normal Distribution		*Gamma Distribution*	
λ_0	γ_0	LM_ρ	$\mathrm{LM}^\star_\rho$	LM_ρ	$\mathrm{LM}^\star_\rho$
-0.30	-0.30	0.223	0.021	0.227	0.017
-0.30	-0.10	0.670	0.125	0.662	0.118
-0.30	-0.05	0.935	0.153	0.934	0.153
-0.30	0.05	1.000	0.105	1.000	0.106
-0.30	0.10	1.000	0.067	1.000	0.065
-0.30	0.30	1.000	0.418	1.000	0.432
-0.10	-0.30	0.046	0.021	0.041	0.021
-0.10	-0.10	0.126	0.042	0.120	0.043
-0.10	-0.05	0.230	0.049	0.234	0.048
-0.10	0.05	0.541	0.045	0.533	0.050
-0.10	0.10	0.638	0.048	0.636	0.043
-0.10	0.30	0.675	0.021	0.670	0.019
-0.05	-0.30	0.043	0.020	0.045	0.020
-0.05	-0.10	0.058	0.039	0.062	0.042
-0.05	-0.05	0.092	0.048	0.094	0.047
-0.05	0.05	0.179	0.050	0.175	0.051
-0.05	0.10	0.221	0.053	0.210	0.049
-0.05	0.30	0.209	0.009	0.208	0.010
0.05	-0.30	0.121	0.024	0.117	0.021
0.05	-0.10	0.065	0.042	0.061	0.041
0.05	-0.05	0.105	0.045	0.114	0.043
0.05	0.05	0.264	0.050	0.274	0.050
0.05	0.10	0.344	0.049	0.364	0.042
0.05	0.30	0.477	0.049	0.484	0.047
0.10	-0.30	0.230	0.032	0.220	0.035
0.10	-0.10	0.157	0.044	0.153	0.042
0.10	-0.05	0.328	0.047	0.326	0.043
0.10	0.05	0.713	0.056	0.732	0.050
0.10	0.10	0.821	0.048	0.821	0.049
0.10	0.30	0.912	0.178	0.918	0.187
0.30	-0.30	0.866	0.028	0.858	0.030
0.30	-0.10	0.783	0.170	0.789	0.170
0.30	-0.05	0.977	0.194	0.974	0.204
0.30	0.05	1.000	0.231	1.000	0.241
0.30	0.10	1.000	0.350	1.000	0.351
0.30	0.30	1.000	1.000	1.000	1.000

For all our proposed tests, the power curves can be generated in several ways. For example, the power curves can be generated when the null model is the 2WE model, and the alternative can be one of the DPD, SSPD, SDPDW and SDPD model. We will refer to this as Case 1. However, this approach would yield several plots, for instance, 216 plots for the 2WE–SDPD combination. To save space, we instead summarize the results in Tables 6 through 8, where the level for all tests is 5%. As we mentioned in the Monte Carlo design, the DPD, SSPD and SDPDW models can be considered as null models for one-directional tests and their robust counterparts. Therefore, we can generate size power curves for these one directional tests, where the null model is one of the DPD, SSPD and SDPDW models and the alternative model is one of the SDPDW and SDPD models. We will refer to this as Case 2. For example, we could investigate the size power curves for LM_λ and $\mathrm{LM}^\star_\lambda$ where the null model is the DPD model and the alternative model is SDPDW model. Similarly, for LM_λ and $\mathrm{LM}^\star_\lambda$, the null of the DPD and the alternative of the SDPD would yield another size power curve. We chose to present some representative cases in Figures 2 and 3.[10]

The general observations from Tables 6 through 8 on the power properties of our proposed tests for Case 1 are listed as follows. To save space, we only present the normally distributed error case, as the results for the gamma distributed error case are similar. Also, for the case of the SDPD model, we focus on some representative tables.

Table 6: Power of tests when H_1: The DPD/SSPD model and H_0: The 2WE model

γ_0/λ_0	LM_ρ	$\mathrm{LM}^\star_\rho$	LM_λ	$\mathrm{LM}^\star_\lambda$	LM_γ	$\mathrm{LM}^\star_\gamma$	LM_J	C_J
			H_1: The DPD model					
-0.30	0.046	0.015	0.042	0.005	1.000	1.000	1.000	1.000
-0.10	0.044	0.038	0.042	0.039	0.550	0.536	0.376	0.374
-0.05	0.040	0.049	0.048	0.051	0.178	0.171	0.114	0.113
0.05	0.061	0.046	0.061	0.056	0.236	0.231	0.149	0.144
0.10	0.074	0.042	0.064	0.039	0.634	0.616	0.454	0.454
0.30	0.135	0.028	0.100	0.024	1.000	1.000	1.000	1.000
			H_1: The SSPD model					
-0.30	1.000	0.791	1.000	1.000	1.000	0.999	1.000	1.000
-0.10	0.913	0.051	0.993	0.810	0.334	0.184	0.975	0.973
-0.05	0.394	0.053	0.600	0.303	0.087	0.071	0.443	0.431
0.05	0.326	0.048	0.593	0.343	0.077	0.068	0.426	0.421
0.10	0.853	0.051	0.992	0.844	0.204	0.136	0.965	0.962
0.30	1.000	0.730	1.000	1.000	0.998	0.997	1.000	1.000

1. Table 6 shows that the joint test statistics and the one directional test statistics, LM_γ, $\mathrm{LM}^\star_\gamma$ in the case of the DPD model and LM_λ, $\mathrm{LM}^\star_\lambda$ in the case of the SSPD model, have desirable power.[11]

2. In Table 6, the robust versions of the one directional tests generally perform similar to their non-robust counterparts. However, as the value of γ_0 increases in the DPD model for example, we see that the rejection frequencies of $\mathrm{LM}^\star_\rho$ remain low whereas LM_ρ over rejects the true null, confirming the (over) size problem in Table 2. A similar finding applies to $\mathrm{LM}^\star_\lambda$. Therefore, in case of temporal dependence in the data generating process, the robust tests are preferable. In the case of the SSPD model in Table 6, $\mathrm{LM}^\star_\gamma$ and $\mathrm{LM}^\star_\rho$ report relatively smaller rejection frequencies and hence perform better than the non-robust counterparts. Again, in case of spatial dependence in the data generating process, the robust tests are preferable.

[10]We only present results based on the rook weight matrix for the power analysis. The results based on the queen weight matrix are available upon request.

[11]Note that the one directional tests and their robust counterparts for λ and ρ should have lower rejection frequencies for the case where H_1: The DPD model and H_0: The 2WE model. Similarly, the one directional tests and their robust counterparts for γ and ρ should report lower rejection frequencies for the case where H_1: The SSPD model and H_0: The 2WE model.

3. Table 7 reveals similar findings. The joint test statistics and the one directional test statistics, LM_γ, $\mathrm{LM}^\star_\gamma$ and LM_λ, $\mathrm{LM}^\star_\lambda$, have desirable power. $\mathrm{LM}^\star_\rho$'s rejection frequency remains low for smaller deviations of λ_0 and γ_0 from zero, whereas LM_ρ over rejects the true null, confirming the (over) size problem in Table 4. Therefore, in case of spatial and temporal dependence in the data generating process, the robust tests are preferable.

4. Tables 8, 9 and 10 shows that all one directional tests and the joint tests have proper power. The non-robust tests have higher power relative to their robust counterparts in some cases but the differences are generally negligible.

For all our proposed tests, the power curves can be generated in several ways in Case 2. First, one can obviously consider the 2WE model as the null model and the alternative can be one of the DPD, SSP, SDPDW and SDPD models. We will not generate size power curves for these cases as we already summarized the results in Tables 6 through 10. Furthermore, for the one directional tests and their robust versions, one of the DPD, SSPD and SDPDW models can be the null model and one of the SDPDW and SDPD models as the alternative model. For example, we can generate a size power curve for LM_λ and $\mathrm{LM}^\star_\lambda$ using the DPD model as the null model and the SDPDW model as the alternative. Another size power curve for LM_λ and $\mathrm{LM}^\star_\lambda$ can be obtained from the DPD model as the null model and the SDPD model as the alternative.[12]

In Figures 2 and 3, the lines with the red color correspond to the non-robust one directional test whereas the lines with blue color correspond to their robust counterparts. Different markers are used to identify varying true values of the spatial parameter in the corresponding alternative model. The general observations on the power properties of our proposed tests are listed as follows.

[12]The experiments based on the gamma distributed errors are not presented, because results are similar to the experiments based on the normally distributed errors.

Table 7: Power of tests when H_1 : The SDPDW model and H_0: The 2WE model

λ_0	γ_0	LM_ρ	$LM^\star_\rho$	LM_λ	$LM^\star_\lambda$	LM_γ	$LM^\star_\gamma$	LM_J	C_J
-0.30	-0.30	0.580	0.299	1.000	1.000	0.979	0.201	1.000	1.000
-0.30	-0.10	0.999	0.833	1.000	1.000	0.964	0.984	1.000	1.000
-0.30	-0.05	1.000	0.831	1.000	1.000	1.000	0.999	1.000	1.000
-0.30	0.05	1.000	0.772	1.000	0.995	1.000	1.000	1.000	1.000
-0.30	0.10	1.000	0.885	1.000	0.992	1.000	1.000	1.000	1.000
-0.30	0.30	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.10	-0.30	0.120	0.045	0.946	0.877	1.000	1.000	1.000	1.000
-0.10	-0.10	0.466	0.039	0.983	0.936	0.274	0.283	0.978	0.980
-0.10	-0.05	0.734	0.046	0.990	0.901	0.088	0.075	0.962	0.963
-0.10	0.05	0.971	0.048	0.995	0.628	0.754	0.525	0.989	0.987
-0.10	0.10	0.990	0.047	0.999	0.483	0.965	0.864	0.998	0.998
-0.10	0.30	1.000	0.739	1.000	0.945	1.000	1.000	1.000	1.000
-0.05	-0.30	0.073	0.025	0.404	0.212	1.000	1.000	1.000	1.000
-0.05	-0.10	0.128	0.044	0.512	0.394	0.481	0.477	0.646	0.663
-0.05	-0.05	0.242	0.050	0.553	0.374	0.128	0.134	0.466	0.473
-0.05	0.05	0.515	0.051	0.651	0.230	0.380	0.296	0.607	0.582
-0.05	0.10	0.612	0.049	0.703	0.162	0.777	0.687	0.827	0.808
-0.05	0.30	0.819	0.219	0.857	0.395	1.000	1.000	1.000	1.000
0.05	-0.30	0.068	0.018	0.457	0.246	1.000	1.000	1.000	1.000
0.05	-0.10	0.121	0.046	0.531	0.411	0.495	0.469	0.652	0.675
0.05	-0.05	0.207	0.049	0.545	0.373	0.154	0.143	0.457	0.473
0.05	0.05	0.474	0.051	0.613	0.244	0.307	0.271	0.566	0.547
0.05	0.10	0.557	0.042	0.629	0.161	0.726	0.685	0.789	0.774
0.05	0.30	0.598	0.022	0.657	0.169	1.000	1.000	1.000	1.000
0.10	-0.30	0.133	0.031	0.949	0.881	1.000	0.999	1.000	1.000
0.10	-0.10	0.360	0.042	0.986	0.948	0.333	0.286	0.979	0.984
0.10	-0.05	0.639	0.053	0.986	0.914	0.082	0.077	0.957	0.961
0.10	0.05	0.956	0.054	0.992	0.706	0.617	0.481	0.984	0.981
0.10	0.10	0.985	0.045	0.995	0.520	0.912	0.828	0.995	0.994
0.10	0.30	0.990	0.151	0.998	0.761	1.000	1.000	1.000	1.000
0.30	-0.30	0.763	0.296	1.000	1.000	0.998	0.228	1.000	1.000
0.30	-0.10	0.976	0.746	1.000	1.000	0.671	0.902	1.000	1.000
0.30	-0.05	1.000	0.757	1.000	1.000	0.961	0.976	1.000	1.000
0.30	0.05	1.000	0.643	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.10	1.000	0.637	1.000	0.999	1.000	1.000	1.000	1.000
0.30	0.30	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 8: Power of tests when H_1:The SDPD model and H_0: The 2WE model

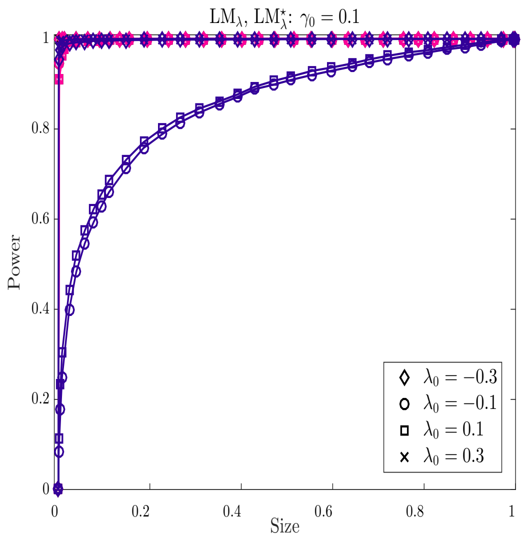
Normal distribution										
λ_0	γ_0	ρ_0	LM_ρ	$LM^\star_\rho$	LM_λ	$LM^\star_\lambda$	LM_γ	$LM^\star_\gamma$	LM_J	C_J
0.05	-0.30	-0.30	1.000	0.964	0.631	0.627	1.000	1.000	1.000	1.000
0.05	-0.30	-0.10	0.861	0.198	0.464	0.020	1.000	1.000	1.000	1.000
0.05	-0.30	-0.05	0.390	0.040	0.430	0.081	1.000	1.000	1.000	1.000
0.05	-0.30	0.05	0.188	0.138	0.478	0.486	1.000	1.000	1.000	1.000
0.05	-0.30	0.10	0.688	0.464	0.496	0.737	1.000	1.000	1.000	1.000
0.05	-0.30	0.30	1.000	0.999	0.503	1.000	1.000	1.000	1.000	1.000
0.05	-0.10	-0.30	1.000	0.991	0.424	0.068	0.929	0.930	1.000	1.000
0.05	-0.10	-0.10	0.446	0.375	0.137	0.167	0.830	0.574	0.796	0.789
0.05	-0.10	-0.05	0.073	0.121	0.287	0.276	0.668	0.509	0.629	0.640
0.05	-0.10	0.05	0.564	0.129	0.758	0.510	0.413	0.486	0.852	0.859
0.05	-0.10	0.10	0.938	0.383	0.917	0.617	0.406	0.533	0.981	0.981
0.05	-0.10	0.30	1.000	0.987	1.000	0.951	0.553	0.891	1.000	1.000
0.05	-0.05	-0.30	1.000	0.991	0.764	0.027	0.547	0.669	1.000	1.000
0.05	-0.05	-0.10	0.292	0.380	0.098	0.220	0.462	0.210	0.530	0.519
0.05	-0.05	-0.05	0.049	0.141	0.262	0.304	0.261	0.156	0.354	0.362
0.05	-0.05	0.05	0.700	0.134	0.824	0.440	0.111	0.141	0.773	0.778
0.05	-0.05	0.10	0.966	0.377	0.964	0.466	0.145	0.185	0.968	0.969
0.05	-0.05	0.30	1.000	0.976	1.000	0.816	0.308	0.588	1.000	1.000
0.05	0.05	-0.30	1.000	0.965	0.997	0.029	0.759	0.098	1.000	1.000
0.05	0.05	-0.10	0.146	0.332	0.069	0.265	0.118	0.166	0.277	0.277
0.05	0.05	-0.05	0.098	0.129	0.221	0.277	0.178	0.235	0.298	0.289
0.05	0.05	0.05	0.883	0.115	0.922	0.186	0.512	0.340	0.876	0.866
0.05	0.05	0.10	0.991	0.322	0.991	0.157	0.633	0.341	0.987	0.986
0.05	0.05	0.30	1.000	0.841	1.000	0.553	0.944	0.225	1.000	1.000
0.05	0.10	-0.30	1.000	0.873	1.000	0.040	0.989	0.490	1.000	1.000
0.05	0.10	-0.10	0.157	0.238	0.084	0.218	0.490	0.497	0.470	0.464
0.05	0.10	-0.05	0.136	0.098	0.204	0.219	0.559	0.610	0.543	0.538
0.05	0.10	0.05	0.929	0.094	0.940	0.129	0.859	0.727	0.964	0.959
0.05	0.10	0.10	0.996	0.218	0.997	0.112	0.931	0.743	0.997	0.996
0.05	0.10	0.30	1.000	0.574	1.000	0.608	0.999	0.712	1.000	1.000
0.05	0.30	-0.30	1.000	0.173	1.000	0.495	1.000	1.000	1.000	1.000
0.05	0.30	-0.10	0.690	0.037	0.368	0.028	1.000	1.000	1.000	1.000
0.05	0.30	-0.05	0.125	0.015	0.120	0.035	1.000	1.000	1.000	1.000
0.05	0.30	0.05	0.985	0.121	0.983	0.507	1.000	1.000	1.000	1.000
0.05	0.30	0.10	1.000	0.330	1.000	0.796	1.000	1.000	1.000	1.000
0.05	0.30	0.30	1.000	0.452	1.000	0.972	1.000	1.000	1.000	1.000

Table 9: Power of tests when H_1: The SDPD model and H_0: The 2WE model

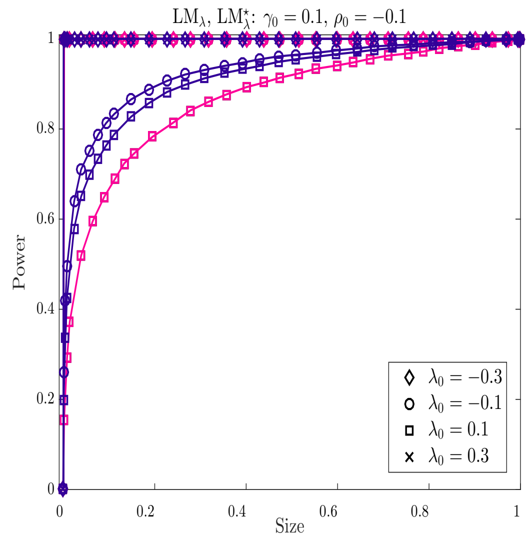
λ_0	γ_0	ρ_0	LM_ρ	$LM_\rho^\star$	LM_λ	$LM_\lambda^\star$	LM_γ	$LM_\gamma^\star$	LM_J	C_J
0.10	-0.30	-0.30	1.000	0.883	0.976	0.137	1.000	1.000	1.000	1.000
0.10	-0.30	-0.10	0.925	0.114	0.954	0.406	1.000	1.000	1.000	1.000
0.10	-0.30	-0.05	0.556	0.016	0.948	0.685	1.000	1.000	1.000	1.000
0.10	-0.30	0.05	0.121	0.191	0.956	0.966	1.000	1.000	1.000	1.000
0.10	-0.30	0.10	0.579	0.574	0.963	0.993	0.999	1.000	1.000	1.000
0.10	-0.30	0.30	1.000	1.000	0.959	1.000	0.995	1.000	1.000	1.000
0.10	-0.10	-0.30	1.000	0.990	0.210	0.077	0.998	0.927	1.000	1.000
0.10	-0.10	-0.10	0.228	0.385	0.815	0.779	0.867	0.405	0.950	0.956
0.10	-0.10	-0.05	0.069	0.154	0.935	0.890	0.619	0.317	0.944	0.952
0.10	-0.10	0.05	0.848	0.113	0.997	0.963	0.176	0.286	0.995	0.995
0.10	-0.10	0.10	0.991	0.340	0.999	0.975	0.159	0.339	1.000	1.000
0.10	-0.10	0.30	1.000	0.975	1.000	0.996	0.336	0.729	1.000	1.000
0.10	-0.05	-0.30	1.000	0.992	0.140	0.138	0.920	0.706	1.000	1.000
0.10	-0.05	-0.10	0.084	0.417	0.758	0.809	0.522	0.117	0.846	0.854
0.10	-0.05	-0.05	0.167	0.168	0.926	0.883	0.218	0.081	0.879	0.887
0.10	-0.05	0.05	0.954	0.101	0.998	0.917	0.133	0.087	0.995	0.995
0.10	-0.05	0.10	0.999	0.308	1.000	0.926	0.250	0.115	1.000	1.000
0.10	-0.05	0.30	1.000	0.928	1.000	0.968	0.647	0.373	1.000	1.000
0.10	0.05	-0.30	0.997	0.989	0.645	0.310	0.271	0.078	0.995	0.994
0.10	0.05	-0.10	0.186	0.353	0.648	0.759	0.087	0.242	0.684	0.672
0.10	0.05	-0.05	0.664	0.150	0.928	0.753	0.290	0.360	0.892	0.880
0.10	0.05	0.05	0.999	0.091	1.000	0.627	0.836	0.560	0.999	0.999
0.10	0.05	0.10	1.000	0.214	1.000	0.554	0.926	0.598	1.000	1.000
0.10	0.05	0.30	1.000	0.503	1.000	0.919	0.998	0.512	1.000	1.000
0.10	0.10	-0.30	0.999	0.972	0.896	0.305	0.782	0.257	0.999	0.998
0.10	0.10	-0.10	0.290	0.250	0.595	0.650	0.400	0.588	0.771	0.757
0.10	0.10	-0.05	0.793	0.094	0.919	0.599	0.685	0.730	0.950	0.942
0.10	0.10	0.05	1.000	0.059	1.000	0.459	0.980	0.872	1.000	1.000
0.10	0.10	0.10	1.000	0.117	1.000	0.474	0.995	0.894	1.000	1.000
0.10	0.10	0.30	1.000	0.183	1.000	0.961	1.000	0.908	1.000	1.000
0.10	0.30	-0.30	1.000	0.206	1.000	0.058	1.000	1.000	1.000	1.000
0.10	0.30	-0.10	0.126	0.019	0.311	0.187	1.000	1.000	1.000	1.000
0.10	0.30	-0.05	0.702	0.028	0.891	0.419	1.000	1.000	1.000	1.000
0.10	0.30	0.05	1.000	0.519	1.000	0.956	1.000	1.000	1.000	1.000
0.10	0.30	0.10	1.000	0.808	1.000	0.994	1.000	1.000	1.000	1.000
0.10	0.30	0.30	1.000	0.877	1.000	1.000	1.000	1.000	1.000	1.000

Table 10: Power of tests when H_1: The SDPD model and H_0: The 2WE model

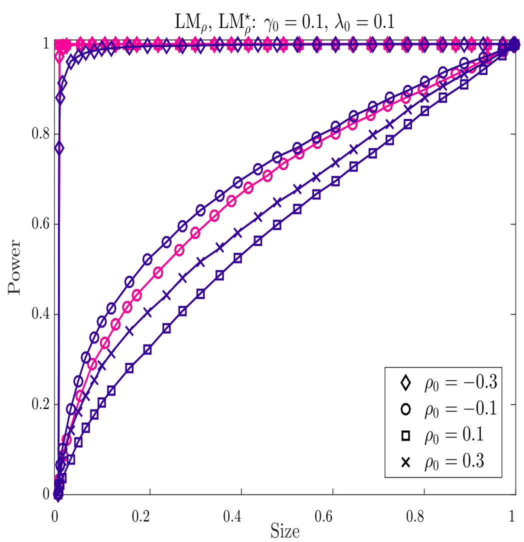
Normal distribution										
λ_0	γ_0	ρ_0	LM_ρ	$LM^\star_\rho$	LM_λ	$LM^\star_\lambda$	LM_γ	$LM^\star_\gamma$	LM_J	C_J
0.30	-0.30	-0.30	1.000	0.766	1.000	0.978	1.000	0.927	1.000	1.000
0.30	-0.30	-0.10	0.999	0.656	1.000	1.000	1.000	0.349	1.000	1.000
0.30	-0.30	-0.05	0.977	0.496	1.000	1.000	1.000	0.269	1.000	1.000
0.30	-0.30	0.05	0.321	0.138	1.000	1.000	0.941	0.233	1.000	1.000
0.30	-0.30	0.10	0.327	0.088	1.000	1.000	0.638	0.235	1.000	1.000
0.30	-0.30	0.30	1.000	0.895	1.000	1.000	0.820	0.450	1.000	1.000
0.30	-0.10	-0.30	1.000	1.000	1.000	1.000	1.000	0.123	1.000	1.000
0.30	-0.10	-0.10	0.296	0.964	1.000	1.000	0.605	0.764	1.000	1.000
0.30	-0.10	-0.05	0.718	0.908	1.000	1.000	0.291	0.865	1.000	1.000
0.30	-0.10	0.05	1.000	0.519	1.000	1.000	0.960	0.913	1.000	1.000
0.30	-0.10	0.10	1.000	0.272	1.000	1.000	0.998	0.920	1.000	1.000
0.30	-0.10	0.30	1.000	0.175	1.000	1.000	1.000	0.892	1.000	1.000
0.30	-0.05	-0.30	0.985	1.000	1.000	1.000	1.000	0.117	1.000	1.000
0.30	-0.05	-0.10	0.734	0.962	1.000	1.000	0.284	0.927	1.000	1.000
0.30	-0.05	-0.05	0.975	0.901	1.000	1.000	0.646	0.967	1.000	1.000
0.30	-0.05	0.05	1.000	0.530	1.000	1.000	0.999	0.982	1.000	1.000
0.30	-0.05	0.10	1.000	0.303	1.000	1.000	1.000	0.983	1.000	1.000
0.30	-0.05	0.30	1.000	0.447	1.000	0.999	1.000	0.985	1.000	1.000
0.30	0.05	-0.30	0.377	0.998	1.000	1.000	0.940	0.373	1.000	1.000
0.30	0.05	-0.10	1.000	0.863	1.000	1.000	0.941	0.998	1.000	1.000
0.30	0.05	-0.05	1.000	0.779	1.000	1.000	0.998	0.999	1.000	1.000
0.30	0.05	0.05	1.000	0.551	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.05	0.10	1.000	0.589	1.000	0.999	1.000	1.000	1.000	1.000
0.30	0.05	0.30	1.000	0.994	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.10	-0.30	0.280	0.988	1.000	1.000	0.531	0.666	1.000	1.000
0.30	0.10	-0.10	1.000	0.659	1.000	1.000	0.998	1.000	1.000	1.000
0.30	0.10	-0.05	1.000	0.621	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.10	0.05	1.000	0.747	1.000	0.999	1.000	1.000	1.000	1.000
0.30	0.10	0.10	1.000	0.891	1.000	0.999	1.000	1.000	1.000	1.000
0.30	0.10	0.30	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.30	-0.30	0.516	0.674	0.999	1.000	1.000	1.000	1.000	1.000
0.30	0.30	-0.10	1.000	0.895	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.30	-0.05	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.30	0.05	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.30	0.10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.30	0.30	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000



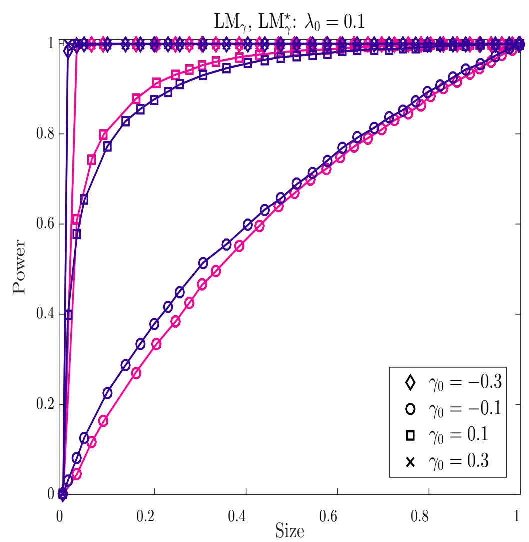
(a) H_0: The DPD model, H_1: The SDPDW model



(b) H_0: The DPD model, H_1: The SDPD model

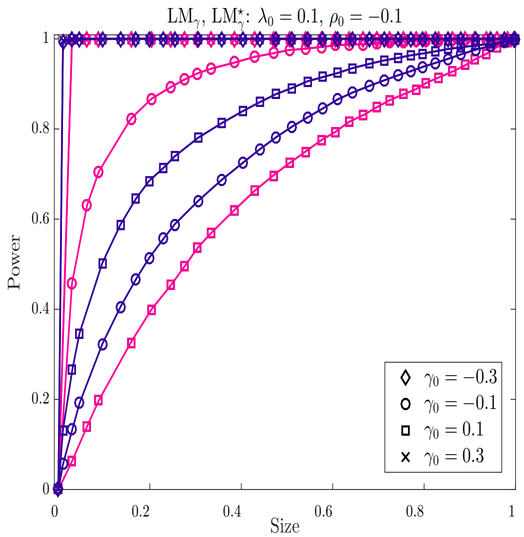


(c) H_0: The DPD model, H_1: The SDPD model

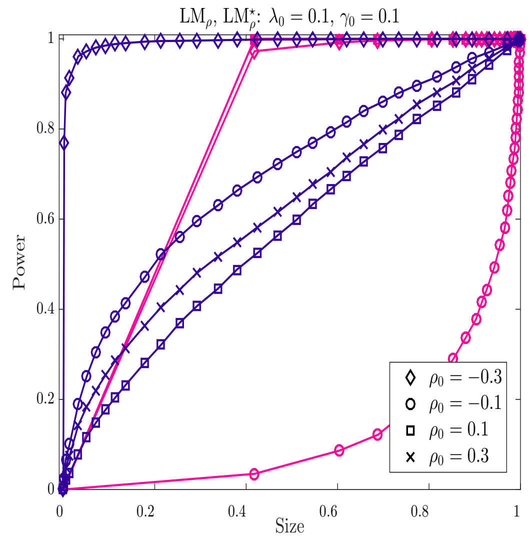


(d) H_0: The SSPD model, H_1: The SDPDW model

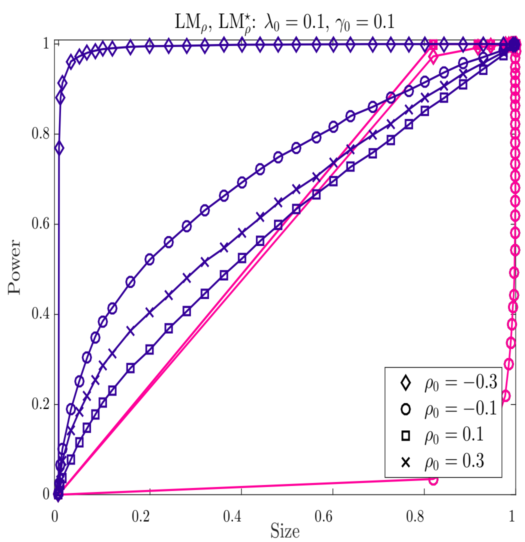
Figure 2: Size-power curves



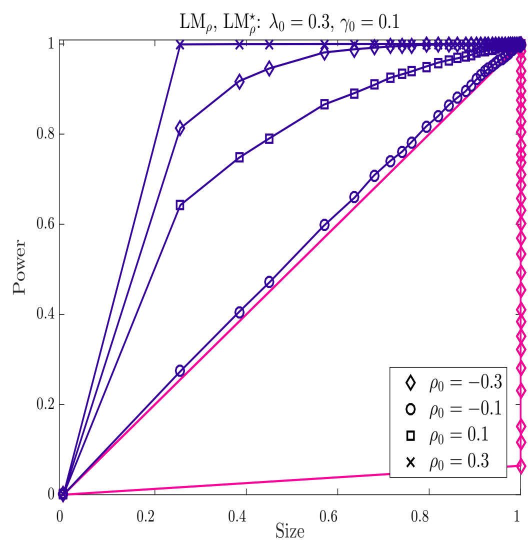
(a) H_0: The SSPD model, H_1: The SDPD model



(b) H_0: The SSPD model, H_1: The SDPD model



(c) H_0: The SDPDW model, H_1: The SDPD model



(d) H_0: The SDPDW model, H_1: The SDPD model

Figure 3: Size-power curves

1. In Figure 2(a), the null model is the DPD model and the alternative model is the SDPDW model. Both LM_λ and $\mathrm{LM}^\star_\lambda$ has satisfactory power. For lower values of λ_0, $\mathrm{LM}^\star_\lambda$ is less powerful than LM_λ. In Figure 2(b), the null model is the DPD model and the alternative model is the SDPD model. Generally, $\mathrm{LM}^\star_\lambda$ is less powerful than LM_λ except for the case where $\lambda_0 = 0.1$.

2. In Figure 2(c), the null model is the DPD model and the alternative model is again the SDPD model. $\mathrm{LM}^\star_\rho$ is slightly less powerful than LM_ρ except for the case where $\rho_0 = -0.1$. In Figure 2(d), the null model is the SSPD model and the alternative model is again the SDPDW model. $\mathrm{LM}^\star_\gamma$ and LM_γ behave similarly and both lack power when $\gamma_0 = -0.1$.

3. In Figure 3(a), the null model is the SSPD model and the alternative model is again the SDPD model. Generally, $\mathrm{LM}^\star_\gamma$ and LM_γ behave similarly. We see that when $\gamma_0 = 0.1$, $\mathrm{LM}^\star_\gamma$ is more powerful than LM_γ. But this picture reverses when $\gamma_0 = -0.1$.

4. In Figure 3(b), the null model is the SSPD model and the alternative model is again the SDPD model. It confirms the results on the one directional tests of ρ_0 from Table 3. LM_ρ over rejects when the true model involves dependence over space and time. Furthermore, when spatial time lag coefficient is small on the negative side, LM_ρ suffers from positive size distortion and lack of power. Surprisingly though, $\mathrm{LM}^\star_\rho$ lack power when $\rho_0 = 0.3$.

5. In Figures 3(c) and 3(d), the null model is the SDPDW model and the alternative model is the SDPD model. It confirms the results on the one directional tests of ρ_0 from Table 4. Clearly, LM_ρ over rejects when the true model involves dependence over space and time. Again, we see that $\mathrm{LM}^\star_\rho$ lack power when $\rho_0 = 0.3$. But, it does not suffer from size distortions unless the misspecification in the alternative becomes larger.

6 Conclusion

In this paper, we introduce the robust LM tests within the GMM framework for a spatial dynamic panel data model. These tests are robust in the sense that their asymptotic distributions under the null hypothesis are still a central chi-square distribution when the alternative model is misspecified. On the other hand, when the alternative model is misspecified, the asymptotic null distributions of the standard LM tests deviate from the central chi-square distributions. Hence, the robust tests obtain asymptotically the correct size. We derive the asymptotic distributions of our proposed tests under the null and the local alternative hypotheses. These tests can be used to test the presence of the contemporaneous dependence over space, dependence over time and spatial time dependence. Since these tests are robust to the misspecification of the alternative models, they are much more suitable for the detection of the source of dependence in a spatial dynamic panel data model.

One attractive feature of our proposed tests is that their test statistics are easy to compute and only require the estimates from a two-way error model. Therefore, our proposed tests can easily be made available for the practical applications by using the standard statistical softwares. In a Monte Carlo study, we investigate the size and power properties of our proposed tests. Our results shows that the robust tests have good finite sample properties and would be useful for the detection of the source of dependence in a spatial dynamic panel data model. The simulation results, hence, confirm our analytical results that the robust tests are valid, when the alternative models locally deviate from the the true data generating process.

Appendix

A A Useful Lemma

Lemma 1. — Under our stated assumptions, the following results hold.

1. $\frac{1}{N}\mathrm{E}\left(g_{nT}(\theta_0)g'_{nT}(\theta_0)\right) = \Sigma_{nT} + o(1)$ and $\widehat{\Sigma}_{nT} = \Sigma_{nT} + o_p(1)$, where $\widehat{\Sigma}_{nT}$ and Σ_{nT} are stated in the main text.
2. $G(\widehat{\theta}_{nT}) = D_{nT} + R_{nT} + O\left(\frac{1}{\sqrt{nT}}\right)$, where D_{nT} is $O(1)$, R_{nT} is $O(\frac{1}{T})$ and $\widehat{\theta}_{nT}$ is any consistent estimator of θ_0.
3. $G(\widehat{\theta}_{nT})\widehat{\Sigma}_{nT}G(\widehat{\theta}_{nT}) = (D_{nT} + R_{nT})' \, \Sigma_{nT} \, (D_{nT} + R_{nT}) + o_p(1)$, where $\widehat{\theta}_{nT}$ is any consistent estimator of θ_0.
4. Let a_{nT} be a $k_a \times (m+q)$ non-stochastic matrix. Then

$$\frac{1}{\sqrt{N}} a_{nT} g_{nT}\left(\theta_0\right) \xrightarrow{d} N\left(0, \mathrm{plim}_{n\to\infty}\, a_{nT}\Sigma_{nT}a'_{nT}\right) \tag{A.1}$$

Proof. See Lee and Yu (2014). □

B Expressions for Test Statistics

In this section, we provide explicit expressions for the elements of test statistics. Let the jth column of $G_a(\theta)$ be denoted by $G_a(\theta)[:,j]$. We start with $G(\theta) = (G_\lambda(\theta), G_\gamma(\theta), G_\rho(\theta), G_\beta(\theta))$, where

$$G_\lambda(\theta)[:,j] = -\frac{1}{N}\begin{pmatrix} \mathbf{Y}^{*'}_{n,T-1}\mathbf{W}'_{nj,T-1}\mathbf{J}_{n,T-1}\mathbf{P}^s_{n1,T-1}\mathbf{J}_{n,T-1}\mathbf{V}^*_{n,T-1}(\theta) \\ \mathbf{Y}^{*'}_{n,T-1}\mathbf{W}'_{nj,T-1}\mathbf{J}_{n,T-1}\mathbf{P}^s_{n2,T-1}\mathbf{J}_{n,T-1}\mathbf{V}^*_{n,T-1}(\theta) \\ \vdots \\ \mathbf{Y}^{*'}_{n,T-1}\mathbf{W}'_{nj,T-1}\mathbf{J}_{n,T-1}\mathbf{P}^s_{nm,T-1}\mathbf{J}_{n,T-1}\mathbf{V}^*_{n,T-1}(\theta) \\ \mathbf{Q}'_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{W}_{nj,T-1}\mathbf{Y}^*_{n,T-1} \end{pmatrix}. \tag{B.1}$$

$$G_\gamma(\theta) = -\frac{1}{N}\begin{pmatrix} \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^s_{n1,T-1}\mathbf{J}_{n,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^s_{n2,T-1}\mathbf{J}_{n,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \\ \vdots \\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^s_{nm,T-1}\mathbf{J}_{n,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \\ \mathbf{Q}'_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \end{pmatrix}. \tag{B.2}$$

$$G_\rho(\theta)[:,j] = -\frac{1}{N}\begin{pmatrix} \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^s_{n1,T-1}\mathbf{J}_{n,T-1}\mathbf{W}_{nj,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^s_{n2,T-1}\mathbf{J}_{n,T-1}\mathbf{W}_{nj,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \\ \vdots \\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^s_{nm,T-1}\mathbf{J}_{n,T-1}\mathbf{W}_{nj,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \\ \mathbf{Q}'_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{W}_{nj,T-1}\mathbf{Y}^{(*,-1)}_{n,T-1} \end{pmatrix}. \tag{B.3}$$

$$G_{\beta}(\theta) = -\frac{1}{N}\begin{pmatrix} \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^{s}_{n1,T-1}\mathbf{J}_{n,T-1}\mathbf{X}^{*}_{n,T-1} \\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^{s}_{n2,T-1}\mathbf{J}_{n,T-1}\mathbf{X}^{*}_{n,T-1} \\ \vdots \\ \mathbf{V}^{*'}_{n,T-1}(\theta)\mathbf{J}_{n,T-1}\mathbf{P}^{s}_{nm,T-1}\mathbf{J}_{n,T-1}\mathbf{X}^{*}_{n,T-1} \\ \mathbf{Q}^{'}_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{X}^{*}_{n,T-1} \end{pmatrix}. \tag{B.4}$$

Using the inverse of the partitioned matrix formula (Amemiya 1985, p.460), we have

$$\begin{aligned} \widehat{\Sigma}^{-1}_{nT} &= \begin{pmatrix} \frac{1}{N}\left[\widehat{\sigma}^{4}\Delta_{nm,T} + \left(\widehat{\mu}_{4} - 3\widehat{\sigma}^{4}\right)\omega^{'}_{nm,T}\omega_{nm,T}\right] & 0_{m\times q} \\ 0_{q\times m} & \widehat{\sigma}^{2}\frac{1}{N}\mathbf{Q}^{'}_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{Q}_{n,T-1} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}, \end{aligned} \tag{B.5}$$

where $O_{11} = N\left[\widehat{\sigma}^{4}\Delta_{nm,T} + \left(\widehat{\mu}_{4} - 3\widehat{\sigma}^{4}\right)\omega^{'}_{nm,T}\omega_{nm,T}\right]^{-1}$, $O_{12} = O^{'}_{21} = 0_{m\times q}$, and $O_{22} = \frac{N}{\widehat{\sigma}^{2}}\left[\mathbf{Q}^{'}_{n,T-1}\mathbf{J}_{n,T-1}\mathbf{Q}_{n,T-1}\right]^{-1}$. The component of $C(\theta)$ are given by

1. $C_{\lambda}(\theta) = G^{'}_{\lambda}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\bar{g}_{nT}(\theta), \quad C_{\gamma}(\theta) = G^{'}_{\gamma}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\bar{g}_{nT}(\theta)$ (B.6)
2. $C_{\rho}(\theta) = G^{'}_{\rho}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\bar{g}_{nT}(\theta), \quad C_{\beta}(\theta) = G^{'}_{\beta}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}\bar{g}_{nT}(\theta)$ (B.7)

The components of $B(\theta)$ are defined in below.

1. $B_{\lambda}(\theta) = G^{'}_{\lambda}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\lambda}, \quad B_{\lambda\rho}(\theta) = B^{'}_{\rho\lambda}(\theta) = G^{'}_{\lambda}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\rho}$
2. $B_{\lambda\gamma}(\theta) = B^{'}_{\gamma\lambda}(\theta) = G^{'}_{\lambda}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\gamma}$, $B_{\lambda\beta}(\theta) = B^{'}_{\beta\lambda}(\theta) = G^{'}_{\lambda}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\beta}$
3. $B_{\rho}(\theta) = G^{'}_{\rho}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\rho}$, $B_{\rho\gamma}(\theta) = B^{'}_{\gamma\rho}(\theta) = G^{'}_{\rho}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\gamma}$
4. $B_{\rho\beta}(\theta) = B^{'}_{\beta\rho}(\theta) = G^{'}_{\rho}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\beta}$, $B_{\gamma}(\theta) = G^{'}_{\gamma}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\gamma}$
5. $B_{\gamma\beta}(\theta) = B^{'}_{\beta\gamma}(\theta) = G^{'}_{\gamma}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\beta}$, $B_{\beta}(\theta) = G^{'}_{\gamma}(\theta)\begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}G_{\beta}$.

Expressions for $H^{\lambda}_{0} : \lambda_{0} = 0$:

$$C^{\star}_{\lambda}(\tilde{\theta}_{nT}) = \left[C_{\lambda}(\tilde{\theta}_{nT}) - B_{\lambda\phi\cdot\beta}(\tilde{\theta}_{nT})B^{-1}_{\phi\cdot\beta}(\tilde{\theta}_{nT})C_{\phi}(\tilde{\theta}_{nT})\right], \tag{B.8}$$

372 where $\phi = \left(\rho^{'}, \gamma\right)^{'}$, $C_{\phi}(\tilde{\theta}_{nT}) = \left(C^{'}_{\rho}(\tilde{\theta}_{nT}), C^{'}_{\gamma}(\tilde{\theta}_{nT})\right)^{'}$, and

$$
\begin{aligned}
B_{\lambda\phi\cdot\beta}(\tilde{\theta}_{nT}) &= B_{\lambda\phi}(\tilde{\theta}_{nT}) - B_{\lambda\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\phi}(\tilde{\theta}_{nT}) \\
&= \left(B_{\lambda\rho}(\tilde{\theta}_{nT}),\, B_{\lambda\gamma}(\tilde{\theta}_{nT})\right) - B_{\lambda\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})\left(B_{\beta\rho}(\tilde{\theta}_{nT}), B_{\beta\gamma}(\tilde{\theta}_{nT})\right)
\end{aligned} \tag{B.9}
$$

$$
\begin{aligned}
B_{\phi\cdot\beta}(\tilde{\theta}_{nT}) &= B_{\phi}(\tilde{\theta}_{nT}) - B_{\phi\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\phi}(\tilde{\theta}_{nT}) \\
&= \begin{bmatrix} B_{\rho}(\tilde{\theta}_{nT}) & B_{\rho\gamma}(\tilde{\theta}_{nT}) \\ B_{\gamma\rho}(\tilde{\theta}_{nT}) & B_{\rho}(\tilde{\theta}_{nT}) \end{bmatrix} - \begin{bmatrix} B_{\rho\beta}(\tilde{\theta}_{nT}) \\ B_{\gamma\beta}(\tilde{\theta}_{nT}) \end{bmatrix} B_{\beta}^{-1}(\tilde{\theta}_{nT}) \left[B_{\beta\rho}(\tilde{\theta}_{nT}),\, B_{\beta\gamma}(\tilde{\theta}_{nT})\right].
\end{aligned} \tag{B.10}
$$

Expressions for $H_0^{\rho}: \rho_0 = 0$:

$$
C_{\rho}^{\star}(\tilde{\theta}_{nT}) = \left[C_{\rho}(\tilde{\theta}_{nT}) - B_{\rho\phi\cdot\beta}(\tilde{\theta}_{nT})B_{\phi\cdot\beta}^{-1}(\tilde{\theta}_{nT})C_{\phi}(\tilde{\theta}_{nT})\right], \tag{B.11}
$$

where $\phi = \left(\lambda',\, \gamma\right)'$, $C_{\phi}(\tilde{\theta}_{nT}) = \left(C_{\lambda}'(\tilde{\theta}_{nT}),\, C_{\gamma}'(\tilde{\theta}_{nT})\right)'$, and

$$
\begin{aligned}
B_{\rho\phi\cdot\beta}(\tilde{\theta}_{nT}) &= B_{\rho\phi}(\tilde{\theta}_{nT}) - B_{\rho\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\phi}(\tilde{\theta}_{nT}) \\
&= \left(B_{\rho\lambda}(\tilde{\theta}_{nT}),\, B_{\rho\gamma}(\tilde{\theta}_{nT})\right) - B_{\rho\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})\left(B_{\beta\lambda}(\tilde{\theta}_{nT}), B_{\beta\gamma}(\tilde{\theta}_{nT})\right),
\end{aligned} \tag{B.12}
$$

$$
\begin{aligned}
B_{\phi\cdot\beta}(\tilde{\theta}_{nT}) &= B_{\phi}(\tilde{\theta}_{nT}) - B_{\phi\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\phi}(\tilde{\theta}_{nT}) \\
&= \begin{bmatrix} B_{\lambda}(\tilde{\theta}_{nT}) & B_{\lambda\gamma}(\tilde{\theta}_{nT}) \\ B_{\gamma\lambda}(\tilde{\theta}_{nT}) & B_{\gamma}(\tilde{\theta}_{nT}) \end{bmatrix} - \begin{bmatrix} B_{\lambda\beta}(\tilde{\theta}_{nT}) \\ B_{\gamma\beta}(\tilde{\theta}_{nT}) \end{bmatrix} B_{\beta}^{-1}(\tilde{\theta}_{nT}) \left[B_{\beta\lambda}(\tilde{\theta}_{nT}),\, B_{\beta\gamma}(\tilde{\theta}_{nT})\right].
\end{aligned} \tag{B.13}
$$

Expressions for $H_0^{\gamma}: \gamma_0 = 0$:

$$
C_{\gamma}^{\star}(\tilde{\theta}_{nT}) = \left[C_{\gamma}(\tilde{\theta}_{nT}) - B_{\gamma\phi\cdot\beta}(\tilde{\theta}_{nT})B_{\phi\cdot\beta}^{-1}(\tilde{\theta}_{nT})C_{\phi}(\tilde{\theta}_{nT})\right], \tag{B.14}
$$

where $\phi = \left(\lambda',\, \rho'\right)'$, $C_{\phi}(\tilde{\theta}_{nT}) = \left(C_{\lambda}'(\tilde{\theta}_{nT}),\, C_{\rho}'(\tilde{\theta}_{nT})\right)'$, and

$$
\begin{aligned}
B_{\gamma\phi\cdot\beta}(\tilde{\theta}_{nT}) &= B_{\gamma\phi}(\tilde{\theta}_{nT}) - B_{\gamma\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\phi}(\tilde{\theta}_{nT}) \\
&= \left(B_{\gamma\lambda}(\tilde{\theta}_{nT}),\, B_{\gamma\rho}(\tilde{\theta}_{nT})\right) - B_{\gamma\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})\left(B_{\beta\lambda}(\tilde{\theta}_{nT}), B_{\beta\rho}(\tilde{\theta}_{nT})\right),
\end{aligned} \tag{B.15}
$$

$$
\begin{aligned}
B_{\phi\cdot\beta}(\tilde{\theta}_{nT}) &= B_{\phi}(\tilde{\theta}_{nT}) - B_{\phi\beta}(\tilde{\theta}_{nT})B_{\beta}^{-1}(\tilde{\theta}_{nT})B_{\beta\phi}(\tilde{\theta}_{nT}) \\
&= \begin{bmatrix} B_{\lambda}(\tilde{\theta}_{nT}) & B_{\lambda\rho}(\tilde{\theta}_{nT}) \\ B_{\rho\lambda}(\tilde{\theta}_{nT}) & B_{\rho}(\tilde{\theta}_{nT}) \end{bmatrix} - \begin{bmatrix} B_{\lambda\beta}(\tilde{\theta}_{nT}) \\ B_{\rho\beta}(\tilde{\theta}_{nT}) \end{bmatrix} B_{\beta}^{-1}(\tilde{\theta}_{nT}) \left[B_{\beta\lambda}(\tilde{\theta}_{nT}),\, B_{\beta\rho}(\tilde{\theta}_{nT})\right].
\end{aligned} \tag{B.16}
$$

C Proofs of Propositions

C.1 Proof of Proposition 1

Let $g_{nT}(\theta)$ denote the $m+q$ dimensional vector of empirical moments such that $m+q \geq 2p+k_x+1$. Define the OGMME $\widehat{\theta}_{nT} = \text{argmin}_{\theta}\, g_{nT}'(\theta)\widetilde{\Sigma}_{nT}^{-1}g_{nT}(\theta)$, where $\widetilde{\Sigma}_{nT}$ is a consistent estimate of Σ_{nT} by Lemma 1. By the implicit function theorem, the set of k_r restrictions on θ_0 can also be stated as $h(\xi_0) = \theta_0$, where $h: \mathbb{R}^{\bar{q}} \to \mathbb{R}^{2p+k_x+1}$ is continuously differentiable, ξ_0 contains the free parameters, and $\bar{q} = 2p + k_x + 1 - k_r$. Define $\widehat{\xi}_{nT} = \text{argmin}_{\xi}\, g_{nT}'\left(h(\xi)\right)\widehat{\Sigma}_{nT}^{-1}g_{nT}\left(h(\xi)\right)$. Then, we have $\widehat{\theta}_{c,nT} =$

$h\left(\widehat{\xi}_{nT}\right)$ as the constrained OGMME of θ_0. Let $\tilde{\xi}_{nT}$ denote a $\sqrt{N}$-consistent estimate of ξ_0.

For notational simplicity, denote $G_\theta = \frac{1}{N}\frac{\partial g_{nT}\left(h(\xi)\right)}{\partial\theta'}$, $\widetilde{G}_\theta = \frac{1}{N}\frac{\partial g_{nT}\left(h(\tilde{\xi}_{nT})\right)}{\partial\theta'}$, $G_\xi = \frac{1}{N}\frac{\partial g_{nT}\left(h(\xi)\right)}{\partial\xi'}$, $\widetilde{G}_\xi = \frac{1}{N}\frac{\partial g_{nT}\left(h(\tilde{\xi}_{nT})\right)}{\partial\xi'}$, and $\tilde{g}_{nT} = g_{nT}\big(h(\tilde{\xi}_{nT})\big)$. By Lemma 1, we have $\text{plim}_{n,T\to\infty}\widetilde{G}_\theta = \mathcal{G}_\theta$, $\text{plim}_{n,T\to\infty}\widetilde{G}_\xi = \mathcal{G}_\xi$, where $\mathcal{G}_\xi = \text{plim}_{n,T\to\infty}\frac{1}{N}\frac{\partial g_{nT}\left(h(\xi_0)\right)}{\partial\xi'}$.

In the following, we first establish the null asymptotic distribution of $C(\alpha)$ test and then that of LM. Our proof for the null asymptotic distribution of $C(\alpha)$ test is similar to the one provided by Lee and Yu (2012b). Let

$$\begin{aligned}
\mathcal{T}^*_{nT}(\xi) &= \frac{1}{N}\frac{\partial g'_{nT}\big(h(\xi)\big)}{\partial\theta}\Bigg[I_{m+q} - \widetilde{\Sigma}^{-1}_{nT}\frac{1}{N}\frac{\partial g_{nT}\big(h(\xi)\big)}{\partial\xi'} \qquad (C.1)\\
&\quad\times\left(\frac{1}{N}\frac{\partial g'_{nT}\big(h(\xi)\big)}{\partial\xi}\widetilde{\Sigma}^{-1}_{nT}\frac{1}{N}\frac{\partial g_{nT}\big(h(\xi)\big)}{\partial\xi'}\right)^{-1}\frac{1}{N}\frac{\partial g'_{nT}\big(h(\xi)\big)}{\partial\xi}\Bigg]\times\widetilde{\Sigma}^{-1}_{nT}\frac{1}{\sqrt{N}}g_{nT}\big(h(\xi)\big)\\
&= G'_\theta\Big[I_{m+q} - \widetilde{\Sigma}^{-1}_{nT}G_\xi\big(G'_\xi\widetilde{\Sigma}^{-1}_{nT}G_\xi\big)^{-1}G'_\xi\Big]\widetilde{\Sigma}^{-1}_{nT}\frac{1}{\sqrt{N}}g_{nT}\big(h(\xi)\big).
\end{aligned}$$

Claim 1. — Let $\mathcal{A}_{nT}$ be any sequence of $(2p+k_x+1)\times\bar{q}$ constant matrices. Define the following class of functions

$$\mathcal{T}_{nT}\big(\mathcal{A}_{nT},\xi\big) = \big(\mathcal{G}'_\theta + \mathcal{A}_{nT}\mathcal{G}'_\xi\big)\Sigma^{-1}_{nT}\frac{1}{\sqrt{N}}g_{nT}\big(h(\xi)\big).$$

Then,

$$\frac{1}{\sqrt{N}}\mathrm{E}\left(\frac{\partial\mathcal{T}_{nT}\big(\mathcal{A}_{nT},\xi_0\big)}{\partial\xi'}\right) = \frac{1}{\sqrt{N}}\mathrm{E}\left(\mathcal{T}_{nT}\big(\mathcal{A}_{nT},\xi_0\big)g'_{nT}(\theta_0)\Sigma^{-1}_{nT}\mathcal{G}_\xi\right) + o(1).$$

Proof. Note that

$$\frac{\partial\mathcal{T}_{nT}\big(\mathcal{A}_{nT},\xi\big)}{\partial\xi'} = \big(\mathcal{G}'_\theta + \mathcal{A}_{nT}\mathcal{G}'_\xi\big)\Sigma^{-1}_{nT}\frac{1}{\sqrt{N}}\frac{\partial g_{nT}\big(h(\xi)\big)}{\partial\xi'}.$$

By Lemma 1, we have

$$\frac{1}{\sqrt{N}}\mathrm{E}\left(\frac{\partial\mathcal{T}_{nT}\big(\mathcal{A}_{nT},\xi_0\big)}{\partial\xi'}\right) = \big(\mathcal{G}'_\theta + \mathcal{A}_{nT}\mathcal{G}'_\xi\big)\Sigma^{-1}_{nT}\mathcal{G}_\xi + o(1).$$

Now, write down

$$\begin{aligned}
\frac{1}{\sqrt{N}}\mathrm{E}\left(\mathcal{T}_{nT}\big(\mathcal{A}_{nT},\xi_0\big)g'_{nT}(\theta_0)\Sigma^{-1}_{nT}\mathcal{G}_\xi\right) &= \big(\mathcal{G}'_\theta + \mathcal{A}_{n,T}\mathcal{G}'_\xi\big)\Sigma^{-1}_{nT}\frac{1}{N}\mathrm{E}\left(g_{nT}\big(h(\xi_0)\big)g'_{nT}(\theta_0)\right)\Sigma^{-1}_{nT}\mathcal{G}_\xi\\
&= \big(\mathcal{G}'_\theta + \mathcal{A}_{n,T}\mathcal{G}'_\xi\big)\Sigma^{-1}_{nT}\mathcal{G}_\xi + o(1), \qquad (C.2)
\end{aligned}$$

where we use the fact that $\frac{1}{N}\mathrm{E}\big(g_{nT}\big(h(\phi_0)\big)g'_{nT}(\theta_0)\big) = \Sigma_{nT} + o(1)$ (see Lemma 1). □

Claim 2. — There exists a unique $\mathcal{A}^*_{nT}$ in the class including $\mathcal{A}_{nT}$ such that

$$\frac{1}{\sqrt{N}}\mathrm{E}\big(\mathcal{T}_{nT}\left(\mathcal{A}^*_{nT},\xi_0\right)g'_{nT}(\theta_0)\Sigma^{-1}_{nT}\mathcal{G}_\xi\big) = o(1),$$

where $\mathcal{A}^*_{nT} = -\mathcal{G}'_\theta\Sigma^{-1}_{nT}\mathcal{G}_\xi\big(\mathcal{G}'_\xi\Sigma^{-1}_{nT}\mathcal{G}_\xi\big)^{-1}$.

Proof. The result follows from setting (C.2) to zero and solving it for $\mathcal{A}_{nT}$. $\square$

Claim 3. — For any $\sqrt{N}$-consistent estimate of $\tilde{\xi}_{nT}$ of ξ_0, we have $\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \tilde{\xi}_{nT}\big) = \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big) + o_p(1)$.

Proof. By assumption $\tilde{\xi}_{nT}$ is a $\sqrt{N}$-consistent estimator. Hence $\sqrt{N}\big(\tilde{\xi}_{nT} - \phi_0\big) = O_p(1)$. By the mean value theorem, we obtain

$$\mathcal{T}_{nT}\big(\mathcal{A}_{nT}, \tilde{\xi}_{nt}\big) = \mathcal{T}_{nT}\big(\mathcal{A}_{nT}, \xi_0\big) + \frac{1}{\sqrt{N}} \frac{\partial \mathcal{T}_{nT}\big(\mathcal{A}_{nT}, \bar{\xi}_{nT}\big)}{\partial \xi'} \sqrt{N}\big(\tilde{\xi}_{nt} - \tilde{\xi}_0\big)$$

where $\bar{\xi}_{nT}$ lies between $\tilde{\xi}_{nt}$ and $\tilde{\xi}_0$. By $\bar{\xi}_{nT} \xrightarrow{p} \xi_0$ and Lemma 1, we obtain

$$\begin{aligned} &\frac{1}{\sqrt{N}} \frac{\partial \mathcal{T}_{nT}\big(\mathcal{A}_{nT}, \bar{\xi}_{nT}\big)}{\partial \xi'} - \frac{1}{\sqrt{N}} \left(\frac{\partial \mathcal{T}_{nT}\big(\mathcal{A}_{nT}, \xi_0\big)}{\partial \xi'} \right) \\ &\qquad\qquad = \big(\mathcal{G}'_\theta + \mathcal{A}_{n,T}\mathcal{G}'_\xi\big)\Sigma^{-1}_{nT} \times \underbrace{\left(\frac{1}{N} \frac{\partial g_{nT}\big(h(\bar{\xi}_{nT})\big)}{\partial \xi'} - \mathcal{G}_\xi \right)}_{o_p(1)} + o_p(1) = o_p(1). \end{aligned}$$

Replacing $\mathcal{A}_{nT}$ with $\mathcal{A}^*_{nT}$ in the mean value expansion and noting from Claim 2 that $\frac{1}{\sqrt{N}} \mathrm{E}\big(\frac{\partial \mathcal{T}_{nT}(\mathcal{A}^*_{nT}, \xi_0)}{\partial \xi'}\big) = o(1)$, we obtain the desired result. $\square$

Claim 4. — At any $\sqrt{N}$-consistent estimate $\tilde{\xi}_{nT}$, $\mathcal{T}^*_{nT}\big(\tilde{\xi}_{nT}\big) - \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \tilde{\xi}_{nT}\big) = o_p(1)$ and $\mathcal{T}^*_{nT}\big(\tilde{\xi}_{nT}\big) = \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big) + o_p(1)$.

Proof. Let $\mathcal{B}_{nT}\big(\tilde{\xi}_{nT}\big) = \widetilde{G}'_\theta\big[\mathbf{I}_{m+q} - \widetilde{\Sigma}^{-1}_{nT}\widetilde{G}_\xi\big(\widetilde{G}'_\xi\widetilde{\Sigma}^{-1}_{nT}\widetilde{G}_\xi\big)^{-1}\widetilde{G}'_\xi\big]\widetilde{\Sigma}^{-1}_{nT}$ and $\mathcal{B}^*_{nT} = \mathcal{G}'_\theta\big[\mathbf{I}_{m+q} - \Sigma^{-1}_{nT}\mathcal{G}_\xi\big(\mathcal{G}'_\xi\Sigma^{-1}_{nT}\mathcal{G}_\xi\big)^{-1}\mathcal{G}'_\xi\big]\Sigma^{-1}_{nT}$. Then, it follows that $\mathcal{T}^*_{nT}\big(\tilde{\xi}_{nT}\big) - \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \tilde{\xi}_{nT}\big) = \big[\mathcal{B}_{nT}\big(\tilde{\xi}_{nT}\big) - \mathcal{B}^*_{nT}\big] \frac{1}{\sqrt{N}} g_{nT}\big(h(\tilde{\xi}_{nT})\big)$. By Lemma 1, $\big[\mathcal{B}_{nT}\big(\tilde{\xi}_{nT}\big) - \mathcal{B}^*_{nT}\big] = o_p(1)$. By the mean value theorem,

$$\begin{aligned} \frac{1}{\sqrt{N}} g_{nT}\big(h(\tilde{\xi}_{nT})\big) &= \frac{1}{\sqrt{N}} g_{nT}\big(h(\xi_0)\big) + \frac{\partial g_{nT}\big(h(\bar{\xi}_{nT})\big)}{\partial \xi'} \frac{1}{\sqrt{N}} \big(\tilde{\xi}_{nT} - \xi_0\big) \\ &= \frac{1}{\sqrt{N}} g_{nT}\big(h(\xi_0)\big) + \frac{1}{N} \frac{\partial g_{nT}\big(h(\bar{\xi}_{nT})\big)}{\partial \xi'} \sqrt{N}\big(\tilde{\xi}_{nT} - \xi_0\big). \end{aligned}$$

Since (i) $\sqrt{N}\big(\tilde{\xi}_{nT} - \xi_0\big) = O_p(1)$, (ii) $\frac{1}{N} \frac{\partial g_{nT}\big(h(\bar{\xi}_{nT})\big)}{\partial \xi'} = \mathcal{G}_\xi + o_p(1)$ by $\bar{\xi}_{nT} \xrightarrow{p} \xi_0$ and Lemma 1, and (iii) $\frac{1}{\sqrt{N}} g_{nT}\big(h(\xi_0)\big) = O_p(1)$, and $\frac{1}{\sqrt{N}} g_{nT}\big(h(\tilde{\xi}_{nT})\big) = O_p(1)$ by Lemma 1. Hence, $\mathcal{T}^*_{nT}\big(\tilde{\xi}_{nT}\big) - \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \tilde{\xi}_{nT}\big) = o_p(1)$. Then, by Claim 3, we have $\mathcal{T}^*_{nT}\big(\tilde{\xi}_{nT}\big) = \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big) + o_p(1)$. $\square$

Claim 5. — Under H_0, the random variable $\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big)$ has zero mean and variance $\Omega = \mathrm{plim}_{n,T\to\infty} \Omega_{nT}$, where $\Omega_{nT} = \mathcal{G}'_\theta\big[\Sigma^{-1}_{nT} - \Sigma^{-1}_{nT}\mathcal{G}_\xi\big(\mathcal{G}'_\xi\Sigma^{-1}_{nT}\mathcal{G}_\xi\big)^{-1}\mathcal{G}'_\xi\Sigma^{-1}_{nT}\big]\mathcal{G}_\theta$ with rank k_r. Furthermore, $\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big) \xrightarrow{d} N\big(0, \Omega\big)$.

Proof. Note that $\mathcal{G}_\theta$ has full rank $2p + k_x + 1$. Hence, $\mathcal{G}'_\theta\Sigma^{-1}_{nT}\mathcal{G}_\theta$ is a positive definite matrix which can be cholesky decomposed as $L_{nT}L'_{nT}$, where L_{nT} is invertible. Further, since $\frac{1}{N}\frac{\partial g_{nT}(h(\xi_0))}{\partial \xi'} =$

$\frac{1}{N}\frac{\partial g_{nT}(\theta_0)}{\partial \theta'}\frac{\partial h(\xi_0)}{\partial \xi'}$, we have $\mathcal{G}_\xi = \mathcal{G}_\theta H_{nT}$, where $H_{nT} = \frac{\partial h(\xi_0)}{\partial \xi'}$. Then, $\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big)$ can be written as

$$\begin{aligned}\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big) &= \big[I_{2p+k_x+1} - \mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\mathcal{G}_\theta H_{nT}\big(H_{nT}^{'}\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\mathcal{G}_\theta H_{nT}\big)^{-1}H_{nT}^{'}\big]\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\frac{1}{\sqrt{N}}g_{nT}(\theta_0) \\ &= L_{nT}\mathbf{M}_{L^{'}H}L_{nT}^{-1}\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\frac{1}{\sqrt{N}}g_{nT}(\theta_0) \end{aligned} \tag{C.3}$$

where $\mathbf{M}_{L^{'}H} = I_{2p+k_x+1} - \mathbf{P}_{L^{'}H}$ and $\mathbf{P}_{L^{'}H} = L_{nT}^{'}H_{nT}\big(H_{nT}^{'}L_{nT}L_{nT}^{'}H_{nT}\big)^{-1}H_{nT}^{'}L_{nT}$. Note that $\mathbf{M}_{L^{'}H}$ is idempotent with its rank equal to $2p + k_x + 1 - \bar{q} = k_r$. Then,

$$\begin{aligned}\mathrm{Var}\big[\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big)\big] &= \mathrm{plim}_{n,T\to\infty} L_{nT}\mathbf{M}_{L^{'}H}L_{nT}^{-1}L_{nT}L_{nT}^{'}L_{nT}^{'-1}\mathbf{M}_{L^{'}H}L_{nT}^{'} \\ &= \mathrm{plim}_{n,T\to\infty} L_{nT}\mathbf{M}_{L^{'}H}L_{nT}^{'} = \mathrm{plim}_{n,T\to\infty}\,\Omega_{nT}\end{aligned}$$

where Ω_{nT} is singular with rank k_r. By Lemma 1, $\frac{1}{\sqrt{N}}g_{nT}(\theta_0) \xrightarrow{d} N\big(0, \mathrm{plim}_{n\to\infty}\Sigma_{nT}\big)$. Hence, $\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big) \xrightarrow{d} N\big(0, \Omega\big)$. □

Claim 6. — Denote $C^*(\alpha) = \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big)^{'}\Omega_{nT}^{-}\mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big)$, where Ω_{nT}^{-} is the generalized inverse of Ω_{nT}.

Proof. It follows from Claim 5 that $C^*(\alpha) \xrightarrow{A} \chi^2_{k_r}$. Note that $\Omega_{nT} = L_{nT}\mathbf{M}_{L^{'}H}L_{nT}^{'}$ and the generalized inverse of $\mathbf{M}_{L^{'}H}$ is itself, then $\Omega_{nT}^{-} = L_{nT}^{'-1}\mathbf{M}_{L^{'}H}L_{nT}^{-1}$. It follows from (C.3)

$$\begin{aligned}C^*(\alpha) &= N\frac{1}{\sqrt{N}}g_{nT}^{'}(\theta_0)\Sigma_{nT}^{-1}\mathcal{G}_\theta L_{nT}^{'-1}\mathbf{M}_{L^{'}H}L_{nT}^{'}\Omega_{nT}^{-}L_{nT}\mathbf{M}_{L^{'}H}L_{nT}^{-1}\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\frac{1}{\sqrt{N}}g_{nT}(\theta_0) \\ &= \frac{1}{N}g_{nT}^{'}(\theta_0)\Sigma_{nT}^{-1}\mathcal{G}_\theta L_{nT}^{'-1}\mathbf{M}_{L^{'}H}L_{nT}^{'}\Omega_{nT}^{-}L_{nT}\mathbf{M}_{L^{'}H}L_{nT}^{'}\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}g_{nT}(\theta_0) \\ &= \frac{1}{N}g_{nT}^{'}(\theta_0)\Sigma_{nT}^{-1}\mathcal{G}_\theta L_{nT}^{'-1}\mathbf{M}_{L^{'}H}L_{nT}^{'}\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}g_{nT}(\theta_0).\end{aligned} \tag{C.4}$$

Note that

$$\begin{aligned}L_{nT}^{'-1}\mathbf{M}_{L^{'}H}L_{nT}^{'} &= \big(L_{nT}L_{nT}^{'}\big)^{-1} - H_{nT}\big(H_{nT}^{'}\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\mathcal{G}_\theta H_{nT}\big)^{-1}H_{nT}^{'} \\ &= \big(\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\mathcal{G}_\theta\big)^{-1} - H_{nT}\big(\mathcal{G}_\xi^{'}\Sigma_{nT}^{-1}\mathcal{G}_\xi\big)^{-1}H_{nT}^{'}\end{aligned} \tag{C.5}$$

Then, it follows from (C.4) and (C.5) that

$$\begin{aligned}C^*(\alpha) = &\frac{1}{N}g_{nT}^{'}(\theta_0)\Sigma_{nT}^{-1}\mathcal{G}_\theta\big(\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}\mathcal{G}_\theta\big)^{-1}\mathcal{G}_\theta^{'}\Sigma_{nT}^{-1}g_{nT}(\theta_0) \\ &- \frac{1}{N}g_{nT}^{'}(\theta_0)\Sigma_{nT}^{-1}\mathcal{G}_\xi\big(\mathcal{G}_\xi^{'}\Sigma_{nT}^{-1}\mathcal{G}_\xi\big)^{-1}\mathcal{G}_\xi^{'}\Sigma_{nT}^{-1}g_{nT}(\theta_0).\end{aligned} \tag{C.6}$$

□

Claim 7. — The test statistic can be written as $C(\alpha) = \mathcal{T}_{nT}\big(\tilde{\xi}_{nT}\big)^{*'}\widetilde{\Omega}_{nT}^{-}\mathcal{T}_{nT}^{*}\big(\tilde{\xi}_{nT}\big)$, where $\widetilde{\Omega}_{nT} = \widetilde{G}_\theta^{'}\big[\widetilde{\Sigma}_{nT}^{-1} - \widetilde{\Sigma}_{nT}^{-1}\widetilde{G}_\xi\big(\widetilde{G}_\xi^{'}\widetilde{\Sigma}_{nT}^{-1}\widetilde{G}_\xi\big)^{-1}\widetilde{G}_\xi^{'}\widetilde{\Sigma}_{nT}^{-1}\big]\widetilde{G}_\theta$. Under H_0, it follows that $C(\alpha) \xrightarrow{d} \chi^2_{k_r}$.

Proof. By Lemma 2, $\widetilde{\Omega}_{nT}^{-} - \Omega_{nT}^{-} = o_p(1)$. Furthermore, by Claim 4 $\mathcal{T}_{nT}^{*}\big(\tilde{\xi}_{nT}\big) = \mathcal{T}_{nT}\big(\mathcal{A}^*_{nT}, \xi_0\big) + o_p(1)$. Hence, $C(\alpha) - C^*(\alpha) = o_p(1)$ by continuous mapping theorem. Then, the asymptotic equivalence (White (2001, Lemma 4.7, p.67)) and Claim 4 yield the desired result. □

Now we will establish the null asymptotic distribution of LM test. Recall that the test statistic is

$$LM = N\, C^{'}\big(\widehat{\theta}_{nT,r}\big) B^{-1}\big(\widehat{\theta}_{nT,r}\big) C\big(\widehat{\theta}_{nT,r}\big). \tag{C.7}$$

Let $\widetilde{LM} = \sqrt{N}\, C^{'}\big(\widehat{\theta}_{nT,r}\big)\mathcal{H}^{-1}\sqrt{N}\, C\big(\widehat{\theta}_{nT,r}\big)$. Under $H_0 : r(\theta_0) = 0$, we have $LM = \widetilde{LM} + o_p(1)$ by Lemma 1 and $\widehat{\theta}_{nT,r} = \theta_0 + o_p(1)$. Now consider the limiting behavior of $\sqrt{N}\, C\big(\widehat{\theta}_{nT,r}\big)$. By the mean value theorem, we have

$$\begin{aligned}\sqrt{N}\, C\big(\widehat{\theta}_{nT,r}\big) &= \sqrt{N}\, C\big(\theta_0\big) - \mathcal{G}^{'}(\bar{\theta})\widehat{\Sigma}_{nT}\mathcal{G}(\bar{\theta}) \times \sqrt{N}\,\big(\widehat{\theta}_{nT,r} - \theta_0\big)\\ &= \sqrt{N}\, C\big(\theta_0\big) - \mathcal{H} \times \sqrt{N}\,\big(\widehat{\theta}_{nT,r} - \theta_0\big) + o_p(1).\end{aligned} \tag{C.8}$$

To evaluate (C.8), we need to consider the limiting behavior of $\sqrt{N}\big(\widehat{\theta}_{nT,r} - \theta_0\big)$. The result derived for the limiting behavior of constrained GMME in Hall (2004, Lemma 5.4, p.167) can be considered for our case. It can be shown that

$$\sqrt{N}\,\big(\widehat{\theta}_{nT,r} - \theta_0\big) = \big[\mathcal{H}^{-1} - \mathcal{H}^{-1}R^{'}\big(R\mathcal{H}^{-1}R^{'}\big)^{-1}R\mathcal{H}^{-1}\big]\sqrt{N}\, C\big(\theta_0\big) + o_p(1), \tag{C.9}$$

where $R = R(\theta_0) = \frac{\partial r(\theta_0)}{\partial \theta^{'}}$. Substituting (C.9) into (C.8) yields

$$\sqrt{N}\, C\big(\widehat{\theta}_{nT,r}\big) = R^{'}\big(R\mathcal{H}^{-1}R^{'}\big)^{-1}R\mathcal{H}^{-1}\sqrt{N}\, C\big(\theta_0\big) + o_p(1). \tag{C.10}$$

Substituting (C.10) into $\widetilde{LM}$ yields $\widetilde{LM} = \sqrt{N}\, C^{'}(\theta_0)\mathcal{H}^{-1}R^{'}\big(R\mathcal{H}^{-1}R^{'}\big)^{-1}R\mathcal{H}^{-1}\sqrt{N}\, C\big(\theta_0\big) + o_p(1)$. By Lemma 1, we have $R\mathcal{H}^{-1}\sqrt{N}\, C\big(\theta_0\big) \xrightarrow{d} N\big(0, R\mathcal{H}^{-1}R^{'}\big)$, which implies that $\widetilde{LM} \xrightarrow{d} \chi^2_{k_r}$. Then, the desired results follows from the asymptotic equivalence of $\widetilde{LM}$ and LM.

C.2 Proof of Proposition 2

The first three results follows directly from $LM_\psi\left(\tilde{\theta}_{nT}\right) \xrightarrow{d} \chi^2_{k_\psi}\,(\vartheta_1)$ under H_A^ψ and H_A^ϕ, where $\vartheta_1 = \delta_\psi^{'}\mathcal{H}_{\psi\cdot\beta}\delta_\psi + \delta_\psi^{'}\mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi + \delta_\phi^{'}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\delta_\psi + \delta_\phi^{'}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\mathcal{H}_{\psi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}\delta_\phi$ is the non-centrality parameter. Here, we will prove the last two results. For this purpose, we consider the distribution of $\mathbf{C}_{\psi\phi}\big(\tilde{\theta}_{nT}\big) = \big(C_\psi^{'}\big(\tilde{\theta}_{nT}\big),\, C_\phi^{'}\big(\tilde{\theta}_{nT}\big)\big)^{'}$. The first order Taylor expansions of $\mathbf{C}_{\psi\phi}\big(\tilde{\theta}_{nT}\big)$ and $C_\beta\big(\tilde{\theta}_{nT}\big)$ around $\theta^* = \big(\beta_0^{'},\, \psi_0^{'} + \delta_\psi^{'}/\sqrt{N},\, \phi_0^{'} + \delta_\phi^{'}/\sqrt{N},\big)^{'}$ are given by

$$\begin{aligned}\sqrt{N}\,\mathbf{C}_{\psi\phi}\big(\tilde{\theta}_{nT}\big) &= \sqrt{N}\,\mathbf{C}_{\psi\phi}\big(\theta^*\big) - \mathbf{G}_{\psi\phi}^{'}\big(\theta^*\big)\widehat{\Sigma}_{nT}^{-1}\mathbf{G}_{\psi\phi}\big(\bar{\theta}\big)\big(\delta_\psi^{'},\, \delta_\phi^{'}\big)^{'}\\ &\quad + \sqrt{N}\,\mathbf{G}_{\psi\phi}^{'}\big(\theta^*\big)\widehat{\Sigma}_{nT}^{-1}G_\beta\big(\bar{\theta}\big)\big(\tilde{\beta}_{nT} - \beta_0\big) + o_p(1),\end{aligned} \tag{C.11}$$

$$\begin{aligned}\sqrt{N}\, C_\beta\big(\tilde{\theta}_{nT}\big) &= \sqrt{N}\, C_\beta\big(\theta^*\big) - G_\beta^{'}\big(\theta^*\big)\widehat{\Sigma}_{nT}^{-1}\mathbf{G}_{\psi\phi}\big(\bar{\theta}\big)\big(\delta_\psi^{'},\, \delta_\phi^{'}\big)^{'}\\ &\quad + \sqrt{N}\, G_\beta^{'}\big(\theta^*\big)\widehat{\Sigma}_{nT}^{-1}G_\beta\big(\bar{\theta}\big)\big(\tilde{\beta}_{nT} - \beta_0\big) + o_p(1),\end{aligned} \tag{C.12}$$

where $\mathbf{G}_{\psi\phi}(\theta) = \big(G_{\psi}(\theta)\,,\, G_{\phi}(\theta)\big)$. Then, using (C.11) and (C.12), we obtain

$$\sqrt{N}\,\mathbf{C}_{\psi\phi}\big(\tilde{\theta}_{nT}\big) = \big(\mathcal{G}_{\psi\phi}^{'}\Sigma_{nT}^{-1},\, \mathbf{H}_{\psi\phi,\beta}\mathcal{H}_{\beta}^{-1}\mathcal{G}_{\beta}^{'}\Sigma_{nT}^{-1}\big)\frac{1}{\sqrt{N}}\,g_{nT}\left(\theta^{*}\right) \\ - \begin{pmatrix} \mathcal{H}_{\psi\cdot\beta} & \mathcal{H}_{\psi\phi\cdot\beta} \\ \mathcal{H}_{\phi\psi\cdot\beta} & \mathcal{H}_{\phi\cdot\beta} \end{pmatrix}\begin{pmatrix} \delta_{\psi} \\ \delta_{\phi} \end{pmatrix} + o_p(1), \tag{C.13}$$

where $\mathcal{G}_{\psi\phi} = \big(\mathcal{G}_{\psi},\, \mathcal{G}_{\phi}\big)$, $\mathbf{H}_{\psi\phi,\beta} = \big(\mathcal{H}_{\psi\beta}^{'},\, \mathcal{H}_{\phi\beta}^{'}\big)^{'}$, and $\mathcal{H}_{\phi\psi\cdot\beta} = \mathcal{H}_{\phi\psi} - \mathcal{H}_{\phi\beta}\mathcal{H}_{\beta}^{-1}\mathcal{H}_{\beta\psi}$. Using Lemma 1, we can determine the distribution of $\sqrt{N}\,\mathbf{C}_{\psi\phi}\big(\tilde{\theta}_{nT}\big)$ under H_0^{ψ} and H_A^{ϕ} from (C.13). Then,

$$\sqrt{N}\,\mathbf{C}_{\psi\phi}\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} N\left(-\begin{pmatrix} \mathcal{H}_{\psi\phi\cdot\beta}\delta_{\phi} \\ \mathcal{H}_{\phi\cdot\beta}\delta_{\phi} \end{pmatrix},\, \begin{pmatrix} \mathcal{H}_{\psi\cdot\beta} & \mathcal{H}_{\psi\phi\cdot\beta} \\ \mathcal{H}_{\phi\psi\cdot\beta} & \mathcal{H}_{\phi\cdot\beta} \end{pmatrix}\right). \tag{C.14}$$

The result in (C.14) can be used to determine the distribution of $\sqrt{N}\,C_{\psi}^{\star}\big(\tilde{\theta}_{nT}\big) = \big(I,\, -\mathcal{H}_{\psi\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\big)\sqrt{N}\,\mathbf{C}_{\psi\phi}\big(\tilde{\theta}_{nT}\big) + o_p(1)$. Hence $\sqrt{N}\,\big[C_{\psi}\big(\tilde{\theta}_{nT}\big) - \mathcal{H}_{\psi\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}C_{\phi}\big(\tilde{\theta}_{nT}\big)\big] \xrightarrow{d} N\big(0, \mathcal{H}_{\psi\cdot\beta} - \mathcal{H}_{\psi\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\phi\psi\cdot\beta}\big)$. Then, this last result and Lemma 1 yield the desired result.

Using (C.13) and Lemma 1, we can also determine the distribution of $\sqrt{N}\,C_{\psi}^{\star}\big(\tilde{\theta}_{nT}\big)$ under H_A^{ψ} and H_0^{ϕ} for the asymptotic power analysis. We have

$$\sqrt{N}\,C_{\phi}^{\star}\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} N\big(-\big(\mathcal{H}_{\psi\cdot\beta} - \mathcal{H}_{\psi\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\big)\delta_{\psi},\, \mathcal{H}_{\psi\cdot\beta} - \mathcal{H}_{\psi\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\big). \tag{C.15}$$

Therefore, $LM_{\psi}^{\star}\big(\tilde{\theta}_{nT}\big) \xrightarrow{d} \chi_{k_{\psi}}^{2}(\vartheta_4)$, where $\vartheta_4 = \delta_{\psi}^{'}\big(\mathcal{H}_{\psi\cdot\beta} - \mathcal{H}_{\psi\phi\cdot\beta}\mathcal{H}_{\phi\cdot\beta}^{-1}\mathcal{H}_{\psi\phi\cdot\beta}^{'}\big)\delta_{\psi}$. It follows that $\vartheta_2 - \vartheta_4 \geq 0$. This result indicates that $LM_{\psi}^{\star}\big(\tilde{\theta}_{nT}\big)$ has less asymptotic power than $LM_{\psi}\big(\tilde{\theta}_{nT}\big)$ when there is no local misspecification.

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